**MATH 220**
**FINAL EXAM**
**Spring 2017**

**NAME :**

**STUDENT I.D. :**

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

*Duration of the examination is 120 minutes, starting at 10:00 and ending at 12:00.*

**GOOD LUCK !**

| 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|
| | | | | |
| 25 | 25 | 25 | 25 | 100 |

**1.a)** Decide whether the following sentences are true or sometimes false. Justify your answer using mathematical arguments (examples, known theorems, or proofs).

**i)** (4 points) If $A$ is a $3 \times 5$ matrix, then the equation $A\mathbf{x} = \mathbf{0}$ has a two-dimensional solution space.

**ii)** (4 points) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set with nonzero vectors, then each vector in the set is expressible as a linear combination of the other two.

**iii)** (4 points) Consider the set $V$ of vectors of the form $(a, b, c, d)$ where $a = d - b$, $b = a - c$, and $d = c + 2b$. Then $V$ is a two-dimensional vector space.

**iv)** (4 points) The set of $2 \times 2$ matrices $M_{22}$ has a basis consisting of invertible matrices.

**1.b)** (9 points) Prove that the formula

$$A \cdot \operatorname{Adj}(A) = \det(A) I_n$$

holds for an arbitrary $n \times n$ matrix $A$.

**2.** Let $L : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by the matrix multiplication $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 & 3 \\ 3 & 6 & -3 & -3 & -3 \\ 2 & 4 & 2 & 0 & 8 \\ 5 & 10 & -2 & -3 & 2 \end{bmatrix}.$$

**a)** (10 points) Convert $A$ into reduced row echelon form.
**b)** (8 points) Find a basis for the kernel of $L$. What is the nullity of $A$?
**b)** (7 points) Find a basis for the range of $L$. What is the rank of $A$?

**3.** Let $P_2$ denote the space of polynomials of degree $\leq 2$, and let

$$S = \{p_1(t), p_2(t), p_3(t)\}$$

be the basis given by $p_1(t) = t^2 - t + 1$, $p_2(t) = t + 2$, $p_3(t) = t^2 + t + 3$.
**a)** (13 points) Use Gram-Schmidt process to transform the basis $S$ into an orthogonal basis $T$ (using standard inner product with respect to standard basis).
**b)** (12 points) Find the change of basis matrix $P_{S \leftarrow T}$.

**4.** Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$.

**a)** (13 points) Find the eigenvalues and associated eigenvectors of $A$. Find a nonsingular matrix $P$ such that $P^{-1}AP = D$ is a diagonal matrix. (For the matrix $P$ that you found, check whether this equation really holds).

**b)** (12 points) Calculate the inverse $A^{-1}$ using row operations. Could we also calculate $A^{-1}$ using the decomposition $A = PDP^{-1}$? Explain your answer.