MATH 220
MIDTERM 2
Fall 2018

NAME : Solutions
STUDENT I.D. :
SECTION :

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 5 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 18:00 and ending at 20:00.

GOOD LUCK !

1	2	3	4	5	Total
20	20	20	25	15	100

1. a) (10 pts) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ a \\ 2 \end{bmatrix}.$$

Find all the values for a so that the vector $\mathbf{v}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. Write the coordinates of $\mathbf{v}$ in with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 = \vec{v} \Rightarrow \lambda_1\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + \lambda_2\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & -1 & a \\ 1 & 4 & 2 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & a \\ 0 & 2 & 2 \end{bmatrix}$$

To be consistant $a = 5$

$$a=5 \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = 1 \end{array}$$

$$[\vec{v}]_{\{\vec{v}_1, \vec{v}_2\}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

b) (10 pts) Let $\mathbf{u} = (1, -1, 2)^T$. The subspace U of $\mathbb{R}^3$ is defined by

$$U = \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} \cdot \mathbf{u} = 0\}$$

Find a basis for U.

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \vec{v}\cdot\vec{u} = 0 \text{ gives } a - b + 2c = 0$$
$$\Rightarrow a = b - 2c$$

$$\Rightarrow \vec{v} = \begin{bmatrix} b-2c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\cdot\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

A Basis for U is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. (20 pts) Consider the matrix

$$A = \begin{bmatrix} -2 & -4 & 1 & 0 & 4 \\ 0 & 0 & 0 & 3 & 3 \\ 1 & 2 & 2 & 1 & 4 \\ 3 & 6 & 0 & -2 & -5 \end{bmatrix}$$

Convert A to the reduced echelon form and answer the following questions.
a) Write down a basis for the column space of A. What is the rank of A?
b) Write down a basis for the subspace consisting of all the solutions to the equation $A\mathbf{x} = 0$.
c) Write down a basis for the subspace consisting of all possible vectors $\mathbf{b}$ such that $A^T\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$.

$$\begin{matrix} \\ +2 \\ -3 \\ \frac{1}{3}\times \end{matrix}\begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ -2 & -4 & 1 & 0 & 4 \\ 3 & 6 & 0 & -2 & -5 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} \longrightarrow \begin{matrix} \\ \\ +3 \\ \\ \end{matrix}\begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 0 & 0 & 5 & 2 & 12 \\ 0 & 0 & -6 & -5 & -17 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 0 & 0 & 5 & 5 & 15 \\ 0 & 0 & -6 & -6 & -18 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon form

a) A basis for column space $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right\}$. $\text{Rank}(A) = 3$

b) Solution space of $A\underline{x} = 0$ is nullspace of A.

$x_2 = s \quad x_5 = t$

$x_1 + 2x_2 - x_5 = 0 \rightsquigarrow x_1 = -2s + t$
$x_3 + 2x_5 = 0 \rightsquigarrow x_3 = -2t$
$x_4 + x_5 = 0 \rightsquigarrow x_4 = -t$

$$\underline{x} = \begin{bmatrix} -2s+t \\ s \\ -2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} t$$

A basis for nullspace $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

c) This is column space of A^T, so it is the same as row space of A.
A basis for row space of A is given by rows of echelon form of A.
So, desired basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

3. (10+10 pts) Let $S = \{u_1, u_2, u_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

a) Applying the Gram-Schmidt process to the basis S find an orthogonal basis $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b) Find the change of basis matrix $P_{S \leftarrow T}$.

a) $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \cdot \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot \vec{v}_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1-1 \\ 3-1-1 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

b)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 1 & 1 \\ -1 & 0 & 3 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & -3 & 0 & 0 & -3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad P_{S \leftarrow T} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Check: $v_1 = u_1$

$v_2 = u_2 - u_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$v_3 = u_3 - u_2 + 2u_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ ✓

4. a) (12 points) Let A be an $m \times n$ matrix where $m \geq n$, and let B be an $n \times n$ matrix such that $AB = 0$. Suppose that the rank of B is r, then what can we say about the rank of A? Justify your answer with mathematical arguments.

$$A \cdot \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = 0 \Rightarrow \text{column of } B \text{ are in the nullspace of } A$$

$\Rightarrow$ Column space of B is a subspace of the nullspace of A

$\Rightarrow$ $r = \text{Rank}(B) \leq \text{Nullity}(A)$

$\Rightarrow \text{Rank}(A) = n - \text{Nullity}(A) \leq n - r$

Since $m \geq n$, $\text{Rank}(A) \leq m$ does not give any more restrictions.

b) (13 points) Let A be an $n \times n$ matrix. Suppose that the product defined on $\mathbb{R}^n$ with the formula $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$ is an inner product. Show that the following two conditions hold: (i) A is invertible (ii) There is an invertible matrix P such that $A = P^T P$.

Let $Aw = 0$. Then $\langle w, w \rangle = w^T \cdot A \cdot w = 0 \Rightarrow w = 0$

So, A is invertible.

$A = A_{\mathcal{E}}$ with respect to standard basis $\mathcal{E} = \{e_1, \ldots, e_n\}$.

If S is another basis, then $\langle \vec{v}, \vec{w} \rangle = [\vec{v}]_S^T \cdot A_S \cdot [\vec{w}]_S = [v]_{\mathcal{E}}^T P_{S \leftarrow \mathcal{E}}^T A_S \cdot P_{S \leftarrow \mathcal{E}} \cdot [w]_{\mathcal{E}}$

Then $A_{\mathcal{E}} = P_{S \leftarrow \mathcal{E}}^T \cdot A_S \cdot P_{S \leftarrow \mathcal{E}}$. If we take S as an orthonormal basis (by Gram-Schmidt such basis always exists), then $A_S = I_n$.

This gives $A = P^T \cdot P$ where $P = P_{S \leftarrow \mathcal{E}}$ (change of basis matrix from standard to an orthonormal basis).

5. For each of the statements below indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument.

(i) (5 pts) Consider the vector space M_{23} of 2×3 matrices. Let W be the subset of M_{23} formed by 2×3 matrices A whose nullity is equal to 1. Then W is a subspace of M_{23}. <u>FALSE</u>

$\underline{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has nullity equal to $3 \neq 1$

so, $\underline{0} \notin M_{23}$. M_{23} can not be a subspace.

(ii) (5 pts) For every nonzero square matrix A, we have $\text{Adj}(A) \neq 0$.

<u>FALSE</u> Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $\text{Adj}(A) = 0$.

(iii) (5 pts) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for a finite dimensional vector space V, then the set of vectors $\{\mathbf{u} - 2\mathbf{v} + 3\mathbf{w}, 2\mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{v} + \mathbf{w}\}$ is also a basis for V.

<u>TRUE</u> $(\vec{u} - 2v + 3w)\lambda_1 + (2u + v - w)\lambda_2 + (u - v + w)\lambda_3 = 0$

$(\lambda_1 + 2\lambda_2 + \lambda_3)\vec{u} + (-2\lambda_1 + \lambda_2 - \lambda_3)\vec{v} + (3\lambda_1 - \lambda_2 + \lambda_3)\vec{w} = 0$

$\{\vec{u}, \vec{v}, \vec{w}\}$ is a basis gives
$$\begin{aligned} \lambda_1 + 2\lambda_2 + \lambda_3 &= 0 \\ -2\lambda_1 + \lambda_2 - \lambda_3 &= 0 \\ 3\lambda_1 - \lambda_2 + \lambda_3 &= 0 \end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & -7 & -2 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ -7 & -2 \end{vmatrix} = -10 + 7 = -3 \neq 0$$

so, $\lambda_1 = \lambda_2 = \lambda_3 = 0$.