MATH 220
MIDTERM I
Fall 2018

NAME : Solutions

STUDENT I.D. :

SECTION :

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 5 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 18:00 and ending at 20:00.

GOOD LUCK !

1	2	3	4	5	Total
20	20	20	25	15	100

1. (20 points) Using Gaussian elimination method (i.e. using augmented matrices) find an equation relating a, b, and c so that the linear system

$$\begin{aligned} 2x + 2y + 3z &= a \\ x - 3y + 2z &= b \\ 3x - y + 5z &= c \end{aligned}$$

is consistent for all values of a, b, and c that satisfy that equation. For $a = b = 1$, $c = 2$, write a parametric solution for this system.

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & a \\ 1 & -3 & 2 & b \\ 3 & -1 & 5 & c \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & b \\ 2 & 2 & 3 & a \\ 3 & -1 & 5 & c \end{array}\right] \xrightarrow[-3R1+R3 \to R3]{-2R1+R2 \to R2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & b \\ 0 & 8 & -1 & a-2b \\ 0 & 8 & -1 & c-3b \end{array}\right]$$

$$\xrightarrow{-R2+R3 \to R3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & b \\ 0 & 8 & -1 & a-2b \\ 0 & 0 & 0 & c-b-a \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & b \\ 0 & 1 & -1/8 & (a-2b)/8 \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

The system is consistent if and only if $c - b - a = 0$

$$\boxed{c = a + b}$$

For $a = b = 1$, $c = 2$, we have

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & -1/8 & -1/8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{z = t}$$

$$x - 3y + 2z = 1$$

$$y - \tfrac{1}{8} z = -\tfrac{1}{8} \Rightarrow \boxed{y = \frac{t-1}{8}}$$

$$x = 3y - 2z + 1$$

$$= 3\left(\frac{t-1}{8}\right) - 2t + 1 = \frac{3t - 3 - 16t + 8}{8} = \frac{5 - 13t}{8}$$

$$\boxed{x = \frac{5 - 13t}{8}}$$

2. (20 points) Let A be a nonsingular 4×4 matrix and P be a 4×4 matrix such that $P \cdot A = R_A$, where R_A is the reduced row echelon matrix which is row equivalent to A. If

$$P \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 - y_2 + y_3 \\ y_4 - y_1 \\ y_4 + y_2 \end{bmatrix}$$

then find A.

Since A is nonsingular, R_A must be identity.

$P \cdot A = I \Rightarrow A = P^{-1}$.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$ We find P^{-1} using row operations.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R1+R3 \\ \to R3}]{\substack{-R1+R2 \\ \to R2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R2 \to R2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\to R4]{-R2+R4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R3 \leftrightarrow R4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R4+R3 \to R3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

3. (20 points) Let A be a 3×4 matrix such that the set of all solutions to the equation $A\underline{x} = \underline{0}$ is given by

$$\underline{x} = t \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

where t, s are any real numbers. Remembering that A is 3×4 matrix, find its reduced row echelon form R.

Since solutions use 2 parameters, we have only 2 colums with leading ones. So A has a zero row.
Parameters t and s are $x_3 = t$, $x_4 = s$
So, columns with no leading 1's are 3 rd and 4th columns

This means A is of the form

$$A = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, $x_1 = t - 2s$ tells us that $\boxed{\begin{aligned} x_1 - x_3 + 2x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}}$

$x_2 = -t - s$

$x_3 = t$

$x_4 = s$

So, $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. a) (13 points) Let $\sigma : X \to X$ be a permutation of the set $\{1, \ldots, n\}$, and let $A(\sigma)$ be the $n \times n$ matrix defined by

$$A(\sigma)_{ij} = \begin{cases} i, & \text{if } j = \sigma(i) \\ 0, & \text{if } j \neq \sigma(i) \end{cases}$$

for all i, j. Calculate the determinant of A in terms of n and σ.

b) (12 points) Calculate the determinant of the matrix $B = A(\sigma_1) - 3A(\sigma_2)$, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

a) $A(\sigma)$ has only one nonzero entry on each row.
On the ith row, the entry $a_{ij} = i$ when $j = \sigma(i)$
$= 0$ otherwise.

So, for example

$$A(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad A(\sigma_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

From the formula for $\det A = \sum_{\sigma \in \text{Perm}\{1,\ldots,n\}} \text{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

we see that for $A(\sigma)$ there is only one term in the sum which is not equal to zero.

So, $\det A = \text{sign}(\sigma) \cdot 1 \cdot 2 \cdots n = \text{sign}(\sigma) \cdot n!$

b) $\det(A(\sigma_1) - 3A(\sigma_3)) = \begin{vmatrix} 1 & -3 & 0 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 3 & -9 \\ -12 & 0 & 0 & 4 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \begin{vmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ -3 & 0 & 0 & 1 \end{vmatrix}$

$= 24 \begin{vmatrix} \boxed{1 \quad 0 \quad 0 \quad 0} \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ -3 & -9 & 0 & 1 \end{vmatrix} = 24 \cdot \begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -3 \\ -9 & 0 & 1 \end{vmatrix} = 24 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ -9 & -27 & 1 \end{vmatrix} = 24(1 - 81)$

$= 24 \cdot (-80)$

$\Rightarrow \boxed{\det B = -1920}$

5. For each of the statements below indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument (for true statements) or give a counterexample (for false statements).

(i) (5 points) Let A and B be $n \times n$ matrices. If AB is invertible, then both A and B are invertible.

TRUE. If AB is invertible, then $\det(A \cdot B) \neq 0$

$\det(A \cdot B) = \det A \cdot \det B$

This gives $\det A \neq 0$ and $\det B \neq 0$

So, both A and B are invertible.

(ii) (5 points) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then there is a nonzero matrix C such that $CA = 0$.

TRUE. $Ax = b$ has infinitely many solutions

$\Rightarrow$ A is not invertible

$\Rightarrow$ Row Echelon form R_A of A has zero row.

So, $\exists\ E_1, \dots, E_k$ such that $E_k \cdots E_1 A = R_A$.

Multiply R_A with $C' = [0\,0 \cdots 0\ 1]$, we get 0.

So, $C = [0 \cdots 0\ 1] \cdot E_k \cdots E_1 \neq 0$ is the desired C.

(iii) (5 points) If A and B are invertible $n \times n$ matrices, then $(A+B)^{-1} = B^{-1} + A^{-1}$.

FALSE. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But

$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.