MATH 220
FINAL EXAM
Fall 2018-2019

NAME :
STUDENT I.D. : Solutions
SECTION :

THIS EXAMINATION PAPER CONTAINS 6 PAGES AND 5 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 18:30 and ending at 20:30.

GOOD LUCK !

1	2	3	4	5	Total
20	25	20	20	15	100

1. a) (10 pts) Let P_3 denote the vector space of all polynomials with degree ≤ 3. Working in the space P_3, find the coordinate vector of $p(x) = x^3 + x^2$ with respect to the ordered basis $S = \{1, (x-1), (x-1)^2, (x-1)^3\}$.

$\lambda_1 \cdot 1 + \lambda_2 (x-1) + \lambda_3 (x-1)^2 + \lambda_3 (x-1)^3 = x^3 + x^2$ gives a system

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -2 & +3 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \text{so, } [p(x)]_S = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 1 \end{bmatrix}.$$

b) (10 pts) Consider the inner product $\langle \mathbf{v}, \mathbf{w} \rangle$ defined on $V = \mathbb{R}^2$ whose matrix with respect to standard basis is equal to $C = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$. Find a basis for V which is orthonormal with respect to the given inner product.

$$\left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle = [a_1 \ a_2] \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 3a_1 b_1 - 2(a_1 b_2 + a_2 b_1) + 3a_2 b_2$$

Let $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Applying the Gram-Schmidt process

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = u_2 - \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{-2}{3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

To get an orthonormal basis, we divide by length.

$$w_1 = \frac{v_1}{\sqrt{\langle v_1, v_1 \rangle}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad w_2 = \frac{v_2}{\sqrt{\langle v_2, v_2 \rangle}} = \frac{1}{\sqrt{3\frac{4}{9} - 4\frac{2}{3} + 3}} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \frac{\sqrt{3}}{\sqrt{5}} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{15} \\ \sqrt{3}/\sqrt{5} \end{bmatrix}$$

2. (25 pts) Consider the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ in $\mathbb{R}^4$ where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -2 \\ -4 \\ 6 \\ 0 \end{bmatrix}.$$

a) Does the set of vectors in S span $\mathbb{R}^4$? If not, describe the subspace $V = span(S)$ in $\mathbb{R}^4$ as the set of vectors $\mathbf{v} = (a, b, c, d)^T$ satisfying certain conditions (linear equations).
b) Find a subset of S that forms a basis for the vector space $V = span(S)$.
c) Let $L : \mathbb{R}_5 \to \mathbb{R}^4$ be the linear transformation defined by

$$L(x_1, x_2, x_3, x_4, x_5) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 + x_5\mathbf{v}_5.$$

Find a basis for the kernel of L.

a)
$$\left[\begin{array}{ccccc|c} \multicolumn{5}{c}{\overbrace{\hphantom{2\;3\;3\;4\;-2}}^{A}} \\ 2 & 3 & 3 & 4 & -2 & a \\ 1 & 3 & 2 & 1 & -4 & b \\ -1 & 0 & -1 & 1 & 6 & c \\ 1 & 3 & 2 & 3 & 0 & d \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 1 & -4 & b \\ 2 & 3 & 3 & 4 & -2 & a \\ -1 & 0 & 1 & -1 & -6 & -c \\ 1 & 3 & 2 & 3 & 0 & d \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 3 & 2 & 1 & -4 & b \\ 0 & -3 & -1 & 2 & 6 & a-2b \\ 0 & -3 & -1 & -2 & -2 & -c-b \\ 0 & 0 & 0 & 2 & 4 & d-b \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 3 & 2 & 1 & -4 & b \\ 0 & -3 & -1 & 2 & 6 & a-2b \\ 0 & 0 & 0 & -4 & -8 & * \\ 0 & 0 & 0 & 2 & 4 & d-b \end{array}\right] \quad -c-b-a+2b$$

We need $\quad -c-b-a+2b = -2(d-b)$

$b-a-c = -2d+2b \Rightarrow \boxed{2d = b+a+c}$

This gives the condition for $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to be in V.

b)
$$\left[\begin{array}{ccccc|c} 1 & 3 & 2 & 1 & -4 & 0 \\ 0 & -3 & -1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} & & s & & t & \\ 1 & 3 & 2 & 1 & -4 & 0 \\ 0 & 1 & 1/3 & -2/3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\{v_1, v_2, v_4\} \subseteq S$ is a basis for V.

c) $\text{Ker}\, L = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1v_1 + \cdots + x_5v_5 = 0\} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} \middle| A\underline{x} = 0 \right\}$

So, Ker L = Nullspace(A).

$\boxed{x_3 = s}$ $\boxed{x_5 = t}$ $\boxed{x_4 = -2t}$

$x_1 = 4t + 2t - 2s - 3\left(-\frac{1}{3}s + \frac{2}{3}t\right)$

$\boxed{x_1 = -s + 4t}$

$x_2 = -\frac{1}{3}x_3 + \frac{2}{3}x_4 + 2x_5$

$= -\frac{1}{3}s - \frac{4}{3}t + 2t = \boxed{-\frac{1}{3}s + \frac{2}{3}t}$

$$\underline{x} = \begin{bmatrix} -1 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 4 \\ 2/3 \\ 0 \\ -2 \\ 1 \end{bmatrix} t$$

Basis for ker L

3. **(20 pts)** Let P_2 denote the vector space of polynomials with degree ≤ 2. Let $L : P_2 \to P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = at^2 + (b + c)t + 2b.$$

Find a basis S for P_2 such that the matrix $A = [L]_{S,S}$ for L associated to the basis S is a diagonal matrix.

With respect to standard basis $\mathcal{E} = \{t^2, t, 1\}$,

$$B = [L]_{\mathcal{E},\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$ because $L(t^2) = t^2$, $L(t) = t + 2$, $L(1) = t$

To diagonalize B, we need to find a basis with eigenvectors for B.

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & -2 & \lambda \end{vmatrix} = (\lambda - 1)[\lambda^2 - \lambda - 2]$$

$$= (\lambda - 1)(\lambda - 2)(\lambda + 1)$$

$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -1$

$\lambda_1 = 1$ $\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -2 & 1 \end{bmatrix} \vec{v}_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$ $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1$ $\quad \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Basis $S = \{t^2, t + 1, -t + 2\}$

4. a) (10 points) Let A be an $n \times n$ matrix whose column vectors form an orthogonal basis for $\mathbb{R}^n$ with respect to the standard inner product. Show that A is invertible.

$$A^T \cdot A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \\ \vdots & & & a_{nn} \end{bmatrix} \begin{bmatrix} \underset{v_1}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}} & \underset{v_2}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ - \end{bmatrix}} & \cdots & \underset{v_n}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \\ & & \ddots & \\ & & & \langle v_n, v_n \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \|v_1\|^2 & 0 & 0 & \cdots & 0 \\ 0 & \|v_2\|^2 & \cdots & & 0 \\ \vdots & & \ddots & & \\ 0 & \cdots & & 0 & \|v_n\|^2 \end{bmatrix} = D$$ where D is diagonal

$\det D = \|v_1\|^2 \cdots \|v_n\|^2 \neq 0$ since $v_1, \ldots, v_n$ all are nonzero vectors.

so, $\det A^T \cdot \det A = \det D \neq 0 \Rightarrow \det A \neq 0$
$\Rightarrow A$ is invertible.

b) (10 points) Let A be an $n \times n$ matrix such that $A_{ij} = 1$ for all i, j. What can we say about the eigenvalues of A? (Hint: First consider the cases $n = 2, 3, 4$ to formulate a claim, then give a general argument for your claim.)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 \\ \vdots & & & & \\ -1 & -1 & \cdots & & \lambda-1 \end{vmatrix} \overset{(*)}{=} \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 \\ \vdots & & & & \\ -1 & \cdots & & \lambda-1 & -1 \\ (\lambda-n) & (\lambda-n) & \cdots & (\lambda-n) & (\lambda-n) \end{vmatrix} \overset{(**)}{=} (\lambda-n) \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & \cdots & -1 \\ & & & \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

(*) add all the rows to the last row (**) factor $(\lambda-n)$ out. (***) add last row to all other rows

$$\overset{(***)}{=} (\lambda-n) \left| \begin{array}{cccc|c} \lambda & 0 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & \vdots \\ & & \ddots & & 0 \\ 0 & \cdots & 0 & \lambda & 0 \\ \hline 1 & 1 & \cdots & & 1 \end{array} \right| = (\lambda - n) \cdot \lambda^{n-1} = 0$$

Eigenvalues are $\lambda = n$ and $\lambda = 0$.

5. For each of the statements below indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument.

(i) (5 pts) Let $M_{m\times n}$ denote the vector space of $m \times n$ matrices over $\mathbb{R}$ with usual addition and multiplication. Let $V \subset M_{m\times n}$ denote the subset consisting of those matrices whose entries all add up to zero. Then V is a subspace of $M_{m\times n}$.

True: If A, B are matrices whose entries all add up to zero, then $A + B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{m}+b_{1n} \\ \vdots & \ddots & \\ & & a_{nn}+b_{nn} \end{bmatrix}$ has all entries add to $(\sum_{i,j} a_{ij}) + (\sum_{i,j} b_{ij}) = 0 + 0 = 0$. Similarly, we can show if $A \in V$, then λA has entry sum $\lambda(\underbrace{\sum a_{ij}}_{0}) = 0$. Also $0 \in V$. So, V is a subspace.

(ii) (5 pts) Let A be an 3×3 matrix which is upper triangular, i.e. $A_{ij} = 0$ when $i > j$. Suppose that A is invertible. Then A^{-1} is also an upper triangular matrix.

True: $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ $\quad$ A is invertible $\Rightarrow \det A = adf \neq 0$.

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} \begin{vmatrix} d & e \\ 0 & f \end{vmatrix} & * & * \\ -\begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & d \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$\boxed{\text{Adj}(A)\text{-formula}}$

So, A^{-1} is upper triangular.

(iii) (5 pts) Let A be a 2×3, and let B be a 3×3 matrix such that $AB = 0$. Suppose that the rank of B is 2. Then the row echelon form of A has a zero row.

True: $A \cdot B = 0 \Rightarrow$ columns of B lies in the nullspace of A

$\Rightarrow$ column space$(B) \subseteq$ Nullspace(A)

$\Rightarrow 2 = rk(B) \leq$ Nullity(A)

$\Rightarrow$ rank$(A) = 3 -$ nullity$(A) \leq 1$.

So, A has at most 1 leading one's in echelon form, so it must have a zero row in its Echelon form.