MATH 220
MIDTERM II
Fall 2019

NAME : Solutions
STUDENT I.D. :
SECTION :

THIS EXAMINATION PAPER CONTAINS 5 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 17:45 and ending at 19:45.

GOOD LUCK !

1	2	3	4	Total
25	25	30	20	100

1. The transpose of a matrix A is the matrix A^T such that $(A^T)_{ij} = A_{ji}$. An $n \times n$-matrix A is called symmetric if $A^T = A$. The trace of an $n \times n$-matrix A is the real number $tr(A) = \sum_{i=1}^{n} A_{ii}$.

a) Consider the set V of all $n \times n$ matrices that are symmetric. Show that V is a vector space under the usual addition and multiplication with scalars defined for matrices. (Hint: You can use the fact that M_{nn} is a vector space under these operations).

b) Let $U \subseteq V$ be the subset of V defined as the set of all matrices $A \in V$ such that A is not invertible. Is U a subspace of V?

c) Let $W \subseteq V$ be the subset of V defined as the set of all matrices $A \in V$ satisfying $tr(A) = 0$. Is W is a subspace of V? If so, for $n = 3$ write a basis for W. What is the dimension of W?

a) It is enough to show that V is a subspace of M_{22}. Note that 0 matrix is symmetric so $V \neq \emptyset$.
If A and B are symmetric, then $(A+B)^T = A^T + B^T = (A+B)$

So, $A+B$ is also sysmmetric.
If $\lambda \in \mathbb{R}$ and A is symmetric, $(\lambda A)^T = \lambda A^T = (\lambda \cdot A)$
$\Rightarrow \lambda A$ is also symmetr.
This shows that V is a vector space.

b) U is <u>not</u> a subspace. Consider
$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both symmetric, <u>not</u> invertible matrices.
But, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric invertible matrix.

So, U is not closed under addition.

c) W is a subspace of U. This is because: $tr(0) = 0$, $0 \in W$.
1) $A, B \in W \Rightarrow tr(A+B) = tr(A) + tr(B) = 0 + 0 = 0$
2) $\lambda \in \mathbb{R}$, $A \in W \Rightarrow tr(\lambda A) = \lambda\, tr(A) = 0$.

For $n=3$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a basis for W.

$\boxed{\mathrm{Dim}(W) = 5}$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & -1 & -4 \\ -2 & -6 & -2 & 1 & 6 \\ 3 & 9 & 0 & -5 & 5 \\ 1 & 3 & -2 & -3 & 13 \end{bmatrix} \text{ where it is given that rref(A)} = \begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) Write down a basis for the column space of A. What is the rank of A?

b) Let W be the space of the solutions to the equation $A\underline{x} = \underline{0}$. Write down a basis for W. What is the dimension of W?

c) Let $W^{\perp}$ denote the orthogonal complement of W with respect to the standard inner product in $\mathbb{R}^5$. Write down a basis for $W^{\perp}$.

a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -5 \\ -3 \end{bmatrix} \right\}$ is a basis for the column space of A.

$\text{Rank}(A) = 3$.

b) $x_2 = s$, $x_5 = t$

$x_1 + 3x_2 + 5x_5 = 0$

$x_3 - 7x_5 = 0$

$x_4 + 2x_5 = 0$

$x_1 = -3s - 5t$

$x_3 = 7t$

$x_4 = -2t$

$$\underline{x} = \begin{bmatrix} -3s-5t \\ s \\ 7t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -5 \\ 0 \\ 7 \\ -2 \\ 1 \end{bmatrix} t$$

Basis for W is $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 7 \\ -2 \\ 1 \end{bmatrix}$

$\text{Dim}(W) = 2$.

c) Row Space of A written as column vectors is the orthogonal complement of W since inner products with rows of A and solutions of $A\underline{x} = \underline{0}$ are zero and it has dimension sum formula to give rows span the orthogonal complement.

From rref(A), we can conclude that

$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $W^{\perp}$.

3. a) Let V be the 3-dimensional space $\mathbb{R}_3$ with the inner product defined by

$$\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1b_1 - a_1b_2 - a_2b_1 + 3a_2b_2 + a_3b_3.$$

Let W be the subspace of V consists of all the vectors (a_1, a_2, a_3) satisfying $a_2 + a_3 = 0$. Find an orthogonal basis for W (with respect to the given inner product) and calculate the projection $\mathrm{Proj}_W \mathbf{v}$ of the vector $\mathbf{v} = (1, -1, 2)$ on the subspace W using the orthogonal basis that you found.

$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ B an orthogonal basis for W.

$$\mathrm{proj}_W \vec{v} = \frac{\langle \vec{v}, \vec{w_1} \rangle}{\langle \vec{w_1}, \vec{w_1} \rangle} \cdot \vec{w_1} + \frac{\langle \vec{v}, \vec{w_2} \rangle}{\langle \vec{w_2}, \vec{w_2} \rangle} \cdot \vec{w_2} = \frac{\left\langle \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\left\langle \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{2}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1 \cancel{+1} \cancel{-1} - 3 + (-2)}{\cancel{1} \cancel{-1} \cancel{-1} + 3 \cancel{+1}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{-4}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 4/3 \end{bmatrix}.$$

b) Let B be a fixed $n \times n$-matrix with rank equal to r. Consider the subspace W of M_{nn} formed by $n \times n$-matrices A satisfying $BA = 0$. What is the dimension of W in terms of n and r? Justify your answer with mathematical arguments.

Columns of A must lie in the nullspace of B.

$\mathrm{Dim}(\mathrm{NullSpace}(B)) = n - r$.

Let $\underline{a}_1, \dots, \underline{a}_{n-r}$ be a basis for nullspace of B. Then,

$$n \begin{cases} [\underline{a}_1 \vdots 0 | 0 \cdots], [\underline{a}_2 \vdots 0 \vdots 0 \cdots], \; \dots, [\underline{a}_{n-r} \vdots 0 | \cdots] \\ [0 \vdots \underline{a}_1 \vdots 0 \cdots], [0 \vdots \underline{a}_2 \vdots \cdots], \; \dots, [0 \vdots \underline{a}_{n-r} \vdots 0 \cdots] \\ \vdots \\ [0 | 0 \cdots | \underline{a}_1], [0 \vdots 0 | \cdots \vdots \underline{a}_2], \; \dots, [0 \vdots \cdots 0 \vdots \underline{a}_{n-r}] \end{cases}$$

is a basis for W. This gives that $\dim W = n \cdot (n-r)$.

4. For each of the statements below indicate whether the statement is always true or sometimes false. If the statement is true, then justify your answer with a logical argument, and if the statement is false, then give an example to illustrate why the statement is false.

(i) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, then the product defined on $\mathbb{R}^2$ with the formula $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A \mathbf{w}$ is an inner product.

FALSE. If we take $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, we find.

$$\left\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\rangle = [-2 \ \ 1]\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0 \quad \text{but} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \neq 0$$

So, this contradicts with inner product property that $\langle \vec{v}, \vec{v} \rangle = 0 \Rightarrow \vec{v} = \vec{0}$.

(ii) The set of vectors $\{f_1, f_2, f_3\}$, where $f_1(x) = x^2 - x + 1$, $f_2(x) = x - 2$, and $f_3(x) = x^2 + 5x + 1$, forms a basis for P_2.

TRUE. It is enough to check the determinant

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 5 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 6 \\ 1 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} = 12 \neq 0.$$

So, $\{f_1, f_2, f_3\}$ is a basis.

(iii) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for $\mathbb{R}^3$, and $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the orthogonal basis obtained from S using the Gram-Schmidt process. Then the change of basis matrix $P_{S \leftarrow T}$ is upper triangular (i.e. $P_{ij} = 0$ for $i > j$).

In the Gram-Schmidt algorithm

TRUE

$$\vec{v}_1 = \vec{u}_1$$
$$\vec{v}_2 = \vec{u}_2 - \alpha \vec{v}_1 = \vec{u}_2 - \alpha \vec{u}_1$$
$$\vec{v}_3 = \vec{u}_3 - \beta \vec{v}_1 - \lambda \vec{v}_2 = \vec{u}_3 - \beta \vec{u}_1 - \lambda(\vec{u}_2 - \alpha \vec{u}_1)$$
$$= \vec{u}_3 + (-\beta + \lambda\alpha)\vec{u}_1 - \lambda \vec{u}_2$$

So, $[\vec{v}_1]_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[\vec{v}_2]_S = \begin{bmatrix} -\alpha \\ 1 \\ 0 \end{bmatrix}$, $[\vec{v}_3] = \begin{bmatrix} -\beta+\lambda\alpha \\ -\lambda \\ 1 \end{bmatrix}$

giving $P_{S \leftarrow T} = \left[[\vec{v}_1]_S \,\vdots\, [\vec{v}_2]_S \,\vdots\, [\vec{v}_3]_S \right] = \begin{bmatrix} 1 & -\alpha & -\beta+\lambda\alpha \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{bmatrix}$ upper triangular.