MATH 220
MIDTERM I
Fall 2019

NAME : Solutions
STUDENT I.D. :
SECTION :

THIS EXAMINATION PAPER CONTAINS 5 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 17:45 and ending at 19:45.

GOOD LUCK !

1	2	3	4	Total
25	30	25	20	100

1. **a)** Using Gaussian elimination method (i.e. using augmented matrices) find an equation relating $a, b,$ and c so that the linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 - 3x_4 &= a \\ -x_1 + 2x_2 - 3x_3 + 4x_4 &= b \\ 3x_1 - x_2 + 4x_3 - 7x_4 &= c \end{aligned}$$

is consistent for all values of $a, b,$ and c that satisfy the equation.

b) Suppose that for some given values of a, b, c you solved the system and found that $x_1 = 4, x_2 = -1, x_3 = 1, x_4 = 2$ is a particular solution for the system. Write all solutions of that system using parameters.

a)
$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & a \\ -1 & 2 & -3 & 4 & b \\ 3 & -1 & 4 & -7 & c \end{array}\right] \xrightarrow[3R1+R3\to R3]{R1+R2\to R2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & a \\ 0 & 1 & -1 & 1 & a+b \\ 0 & 2 & -2 & 2 & c-3a \end{array}\right] \xrightarrow{-2R2+R3 \to R3} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & a \\ 0 & 1 & -1 & 1 & a+b \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

$$* = (c-3a) - 2(a+b) = \boxed{c - 5a - 2b = 0}$$

condition for the system to be consistent

b) For $x_1 = 4, x_2 = -1, x_3 = 1, x_4 = 2$, we find

$a = 4 + 1 + 2 - 3 \cdot 2 = 1$

$b = -4 + 2(-1) - 3 + 8 = -1$

$c = 12 + 1 + 4 - 14 = 3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R2+R1 \to R1} \begin{array}{c} \begin{array}{cccc} & & s & t \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Parametric solution:
$$\boxed{\begin{aligned} x_1 &= 1 - s + 2t \\ x_2 &= s - t \\ x_3 &= s \\ x_4 &= t \end{aligned}} \qquad t, s \in \mathbb{R}$$

<u>Check</u>: for $s = 1$, $t = 2$, $x_1 = 4$, $x_2 = -1$, $x_3 = 1$, $x_4 = 2$ is the given particular solution.

2. If it exists, find A^{-1} for

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 0 & -4 \\ 1 & -1 & 2 \end{bmatrix}$$

a) Using the augmented matrix and row operations.

b) Using the inverse matrix formula involving the adjoint matrix formed by cofactors.

a)
$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & -4 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R1+R3\to R3]{2R1+R2\to R2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{4R3+R2\to R2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & 4 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R2\leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & -2 & 1 & 4 \end{array}\right] \frac{1}{2}\times$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{3}{2} & -4 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & 2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 2 & -3/2 & -4 \\ 0 & -1/2 & -1 \\ -1 & 1/2 & 2 \end{bmatrix}$$

b)
$$\det(A) = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 0 & -4 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & -4 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -4+2 = -2 \neq 0.$$

$C_{11} = \begin{vmatrix} 0 & -4 \\ -1 & 2 \end{vmatrix} = -4 \qquad C_{21} = -\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} = 3 \qquad C_{31} = \begin{vmatrix} -2 & 1 \\ 0 & -4 \end{vmatrix} = 8$

$C_{12} = -\begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = 0 \qquad C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \qquad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = 2$

$C_{13} = \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} = 2 \qquad C_{23} = -\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 \qquad C_{33} = \begin{vmatrix} 1 & -2 \\ -2 & 0 \end{vmatrix} = -4$

$$A^{-1} = \frac{1}{\det A} \cdot \mathrm{Adj}(A) = \frac{1}{-2} \begin{bmatrix} -4 & 3 & 8 \\ 0 & 1 & 2 \\ 2 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 & -4 \\ 0 & -1/2 & -1 \\ -1 & 1/2 & 2 \end{bmatrix}$$

=

3. a) Calculate the determinant

$$\begin{vmatrix} 1 & -1 & a & 2 \\ -3 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & b & 2 & 1 \end{vmatrix}$$

in terms of a and b.

$$\begin{vmatrix} 1 & -1 & a & 2 \\ -3 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & b & 2 & 1 \end{vmatrix} \overset{+1}{=} \begin{vmatrix} 1 & -1 & (a-1) & 2 \\ -3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & b & (2+b) & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a-1 & 2 \\ -3 & 2 & 1 \\ -1 & 2+b & 1 \end{vmatrix} \quad {+3}\quad {-2}$$

$$= -\begin{vmatrix} 7 & a-5 & 2 \\ 0 & 0 & 1 \\ 2 & b & 1 \end{vmatrix} = \begin{vmatrix} 7 & a-5 \\ 2 & b \end{vmatrix} = 7b - 2a + 10$$

b) Let $A = (a_{ij})$ be an $n \times n$ matrix such that

$$a_{ij} = \begin{cases} 1, & \text{if } i+j = n+1 \\ 0, & \text{if } i+j \neq n+1 \end{cases}$$

for all i, j. Calculate the determinant of A in terms of n.

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots 1 & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{bmatrix}_{n \times n}$$. We have $\det(A) = \text{sign}(\sigma)$ where $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & n \\ n & n-1 & n-2 & \cdots & n-i & \cdots & 1 \end{pmatrix}$.

If $\overset{\text{even}}{n = 2k}$, then $\text{sign}(\sigma) = (-1)^k$ because then we have exactly k many involutions (swaps) to go to the identity.

Ex $n=4$ $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ 4321 → 4231 → 1234.

If $\overset{\text{odd}}{n = 2k+1}$, then $\text{sign}(\sigma) = (-1)^k$ because we again need k many swaps.

Ex $n=5 = 2\cdot 2+1$ $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ 54321. 2 swaps

$n = 7 = 2\cdot 3+1$ 7654321 3 swaps

So, $\boxed{\det(A) = (-1)^k \text{ when } n = 2k \text{ or } = 2k+1}$

4. For each of the statements below indicate whether the statement is always true or sometimes false. Justify your answer with a logical argument (for true statements) or give a counterexample (for false statements).

(i) Let A be an $n \times n$ matrix for some $n \geq 1$. If two of the columns of A are equal to each other, then the homogeneous system $A\underline{x} = \underline{0}$ has infinitely many solutions.

TRUE

If two columns are equal than $\det(A) = 0$.
Then, A is not invertible and $\mathrm{rref}(A)$ has a zero row
This gives that $A\underline{x} = \underline{0}$ has infinitely many solutions.

(ii) Let A be a 3×3 matrix. If $\det(A) = 2$ then A has a row such that all the entries on that row are divisible by 2.

FALSE.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

has $\det(A) = 2$
but does not have a row with entries divisible by 2.

(iii) Let A be a 3×4 matrix. If for some 3×1 matrix $\underline{b}$ the system $A\underline{x} = \underline{b}$ has two distinct solutions, then the reduced row echelon from of A has a zero row.

FALSE

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

has infinitely many solutions for any $\underline{b}$
but does not have a zero row.