Math 220 - Linear Algebra
Homework 4 - Solutions

1 a)

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & a \\ 3 & 1 & 1 & -1 & 3 & b \\ 4 & 1 & 2 & -2 & -5 & c \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & a \\ 0 & 1 & -2 & -4 & 3 & b-3a \\ 0 & 1 & -2 & -6 & -5 & c-4a \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 0 & a \\ 0 & 1 & -2 & -4 & 3 & b-3a \\ 0 & 0 & 0 & -2 & -8 & c-b-a \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & \boxed{0} & 1 & \boxed{1} & 0 & a \\ \boxed{0} & \boxed{1} & -2 & \boxed{-4} & 3 & b-3a \\ \boxed{0} & \boxed{0} & 0 & \boxed{1} & 4 & \frac{\text{[illegible]}}{2} \end{array}\right]$$

No zero row. So, for every a, b, c, there is a solution to the above system. $\{f_1, .., f_5\}$ spans $V = P_2$.

b) $\{f_1, f_2, f_4\}$ is a basis for $\text{Span}\{f_1, .., f_5\}$ because the corresponding columns have a leading one.

c)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & 1 & -1 & 3 \\ 4 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -6 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -2 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \Rightarrow [\vec{v}]_{S'} = \begin{bmatrix} -1/2 \\ 6 \\ 3/2 \end{bmatrix}.$$

2

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ -1 & -2 & 0 & 2 & 2 \\ 0 & -3 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & -3 & 2 & 0 & 1 \\ 0 & -2 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 7 \\ 0 & 0 & -1 & 3 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -4 & -7 \\ 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 8 \\ 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2

a) $\{[1\ 0\ 0\ 2\ 8], [0\ 1\ 0\ -2\ -5], [0\ 0\ 1\ -3\ -7]\}$ is a basis for the row space of A. $\text{RowRank}(A) = 3$.

b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis for the column space of A.

$\text{ColumnRank}(A) = 3$.

c) $\left.\begin{aligned} x_1 + 2x_4 + 8x_5 &= 0 \\ x_2 - 2x_4 - 5x_5 &= 0 \\ x_3 - 3x_4 - 7x_5 &= 0 \end{aligned}\right\}$ $\quad x_4 = s \quad x_5 = t \quad$ $\begin{aligned} x_1 &= -2s - 8t \\ x_2 &= 2s + 5t \\ x_3 &= 3s + 7t \end{aligned}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-8t \\ 2s+5t \\ 3s+7t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v_1}} s + \underbrace{\begin{bmatrix} -8 \\ 5 \\ 7 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v_2}} t$$

$\{\vec{v_1}, \vec{v_2}\}$ is a basis for $\text{NullSpace}(A)$.

$\text{Nullity}(A) = 2$.

3

$$\begin{array}{cccccc} v_1 & v_2 & v_3 & \vec{w_1} & \vec{w_2} & \vec{w_3} \end{array}$$

$$\left[\underbrace{\ \vdots\ \ \vdots\ \ \vdots\ }_{M_S} \Big| \underbrace{\ \vdots\ \ \vdots\ \ \vdots\ }_{M_T}\right] \to \cdots \to [\, I \mid P_{S \leftarrow T} \,]$$

So, $P_{S \leftarrow T} = M_S^{-1} \cdot M_T$

$\Rightarrow M_T = M_S \cdot P_{S \leftarrow T}$

$$M_T = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -9 & 2 & 2 \\ -1 & 0 & 1 \\ 9 & -1 & -4 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{w_1} & \vec{w_2} & \vec{w_3} \end{array}$$

(4) $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \cdot \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{\overbrace{\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle}^{-2}}{\underbrace{\left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle}_{3}} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$

$$\rightsquigarrow \vec{v}_2' = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \cdot \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \cdot \vec{v}_2$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \frac{4}{3} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{(-1)}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4/3 \\ 4/3 \\ -4/3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 + 3/2 \\ 1 - 3/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix} \rightsquigarrow \vec{v}_3' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{w}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{w}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

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(5) a) TRUE

Let V be a vector space and $S_1 \subseteq S_2$ two sets of vectors in V.

Let $S_1 = \{\vec{v}_1, \dots, \vec{v}_k\}$

$S_2 = \{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$.

We know S_1 spans V.

Take $\vec{v} \in V$. Then there is $\lambda_1, -, \lambda_k$ such that

$$\vec{v} = \lambda_1 \vec{v_1} + - - + \lambda_k \vec{v_k}$$
$$= \lambda_1 \vec{v_1} + - - + \lambda_k \vec{v_k} + 0 \cdot \vec{v_{k+1}} + - - - + 0 \cdot \vec{v_n}$$

so, $\vec{v} \in \text{Span}\{\vec{v_1}, -, \vec{v_n}\}$. This show S_2 spans V.

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b) TRUE

$$[Av_1 \mid - - \mid Av_n] = A \cdot \underbrace{[\vec{v_1} \mid - - \mid \vec{v_n}]}_{B}$$

Since $\{\vec{v_1}, -, \vec{v_n}\}$ is a basis, $\det(B) \neq 0$.

It is also given that $\det(A) \neq 0$.

so, $\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0$

This gives that $\{A\vec{v_1}, -, A\vec{v_n}\}$ is a basis for V.

c) FALSE.

Take $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}_{3\times 3}$

$A \cdot B = 0_{2\times 3}$ but $\text{rank}(A) = 2$

not strictly smaller than 2.
