Math 220 - HW3 Solutions - Fall 2019

(1) a) The set of triples (x,y,z) with the operations
$(x,y,z)+(x',y',z')=(x+x',y+y',z+z')$ and
$\lambda(x,y,z)=(\lambda x,y,z)$ is not a vector space.
The axiom $(\lambda+\beta)\vec{v}=\lambda\vec{v}+\beta\vec{v}$ does not hold.

$$(\lambda+\beta)(x,y,z)=(\lambda x+\beta x,\ y,z) \quad \text{not equal}$$
$$\lambda(x,y,z)+\beta(x,y,z)=(\lambda x,y,z)+(\beta x,y,z)=(\lambda x+\beta x,2y,2z)$$

b) The set of pairs $(1,y)$ with
$(1,y)+(1,y')=(1,y+y')$ and $\lambda(1,y)=(1,\lambda y)$.
This is a vector space, it satisfies all the axioms.
The zero vector is $(1,0)$ and inverse of $(1,y)$ is $(1,-y)$

c) The set of real valued functions satisfying $f(1)=0$.
The operations are
$$(f+g)(x)=f(x)+g(x), \quad (\lambda f)(x)=\lambda f(x).$$
One can check the axioms one by one to see that this is a vector space. Alternatively observe $W=\{f \mid f(1)=0\}$ is a subspace of all functions.
To see this
$$(f+g)(1)=f(1)+g(1)=0+0=0$$
$$(\lambda f)(1)=\lambda f(1)=\lambda\cdot 0=0$$
From this we can conclude that W is a vector space

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(2) M_{22} denotes the vector space of 2×2 matrices with real number entries.

a) $W=\left\{\begin{bmatrix} a & a+b \\ a+b & b\end{bmatrix} \mid a,b\in\mathbb{R}\right\}\subseteq M_{22}$

1) $\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} + \begin{bmatrix} a' & a'+b' \\ a'+b' & b' \end{bmatrix} = \begin{bmatrix} (a+a') & (a+a')+(b+b') \\ (a+a')+(b+b') & b+b' \end{bmatrix} \in W$

2) $\lambda \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda a+\lambda b \\ \lambda a+\lambda b & \lambda b \end{bmatrix} \in W$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ so $W \neq \emptyset$. Hence W is a subspace.

b) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a^2+b=d \right\}$ is <u>not</u> a subspace

Take $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W$

or $3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \notin W$.

So, both axioms for a subspace fail.

c) $W = \{ A \in M_{22} \mid \det A = 0 \}$.

Since $\det(A+B) \neq \det A + \det B$, W will not be a subspace.

Take $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$.

Then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$ since its determinant is $1 \neq 0$.

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3 a) $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \to \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 3 & -1 & 2 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 5 & 2 & -7 \\ 0 & 5 & 1 & -6 \end{bmatrix} \to \begin{matrix} \\ \frac{1}{5} \\ \frac{1}{3}\times \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

$\to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ has a solution. So, $\vec{v}$ is in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

b) $\det\begin{bmatrix}3 & 2 & 4\\ 1 & -1 & 0\\ 1 & 5 & -3\end{bmatrix} \overset{+1}{=} \begin{vmatrix}3 & 5 & 4\\ 1 & 0 & 0\\ 1 & 6 & -3\end{vmatrix} = -\begin{vmatrix}5 & 4\\ 6 & -3\end{vmatrix} = -(-15-24)$

$= 39 \neq 0$

So, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

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a) $\left[\begin{array}{ccc|c}1 & 2 & 3 & 2\\ 0 & 2 & 3 & -1\\ 0 & 0 & 3 & 3\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\ 0 & 2 & 0 & -4\\ 0 & 0 & 3 & 3\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 1\end{array}\right]$

So, $v = 3\vec{v}_1 + (-2)\vec{v}_2 + \vec{v}_3$ coordinates are $[\vec{v}]_S = \begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$.

b) $\left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 2 & 5 & -8 & -12\\ 3 & 6 & 9 & 3\end{array}\right] \to \left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 0 & 13 & -22 & -22\\ 0 & 18 & -12 & -12\end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 0 & 13 & -22 & -22\\ 0 & 5 & 10 & 10\end{array}\right] \frac{1}{5}$

$\to \left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 0 & 1 & 2 & 2\\ 0 & 13 & -22 & -22\end{array}\right] \to \left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 0 & 1 & 2 & 2\\ 0 & 0 & -48 & -48\end{array}\right] \to \left[\begin{array}{ccc|c}1 & -4 & 7 & 5\\ 0 & 1 & 2 & 2\\ 0 & 0 & 1 & 1\end{array}\right]$

$\to \left[\begin{array}{ccc|c}1 & -4 & 0 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 0 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{array}\right]$, $[\vec{v}]_S = \begin{bmatrix}-2\\ 0\\ 1\end{bmatrix}$.
