MATH 220
FINAL EXAM
Fall 2019-2020

NAME :
STUDENT I.D. : Solutions
SECTION :

THIS EXAMINATION PAPER CONTAINS 5 PAGES AND 4 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. PLEASE BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

No books, notes or calculators of any kind are allowed. You must give detailed mathematical explanations for all your conclusions in order to receive full credit.

Duration of the examination is 120 minutes, starting at 15:30 and ending at 17:30.

GOOD LUCK !

1	2	3	4	Total
25	30	25	20	100

1. Let P_n denote the vector space of polynomials with degree $\leq n$. Consider the linear transformation $L : P_3 \to P_2$ defined by

$$L(p(t)) = (t+1)p(t)' - 3p(t)$$

a) Find the matrix $A = [L]_{S,T}$ for the linear transformation L with respect to standard basis $S = \{t^3, t^2, t, 1\}$ and $T = \{t^2, t, 1\}$.
b) Convert A into the reduced row echelon form and find a basis for the kernel of L. What can you say about the image of L?

a) $L(t^3) = (t+1)3t^2 - 3t^3 = 3t^2$

$L(t^2) = (t+1)2t - 3t^2 = -t^2 + 2t$

$L(t) = (t+1)\cdot 1 - 3t = -2t + 1$

$L(1) = -3$

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

b) $\begin{matrix} \frac{1}{3}\times \\ \frac{1}{2}\times \\ \ \end{matrix}\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \xrightarrow{+\frac{1}{3}} \begin{bmatrix} 1 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

$\to \begin{bmatrix} (1) & 0 & 0 & -1 \\ 0 & (1) & 0 & -3 \\ 0 & 0 & (1) & -3 \end{bmatrix}$ ← s

$x_4 = s$

$x_1 - x_4 = 0 \Rightarrow x_1 = s$

$x_2 - 3x_4 = 0 \Rightarrow x_2 = 3s$

$x_3 - 3x_4 = 0 \Rightarrow x_3 = 3t$

$\underline{x} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} s$ solution space

A basis for kernel is $\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$ or $\{p(t) = (t+1)^3\}$. Thus nullity$(A) = 1$.

This means rank$(A) = 3$, giving that L is onto, so the image is P_2.

(because rank(A) + Nullity$(A) = 4$)

2. Consider the matrix $A = \begin{bmatrix} -3 & 4 & 6 \\ -2 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}$.

a) Find the eigenvalues and associated eigenvectors of A. Find a nonsingular matrix P such that $P^{-1}AP = D$ is the diagonal matrix with diagonal entries given by eigenvalues.

b) Calculate the inverse matrix P^{-1} and check whether or not the equation $P^{-1}AP = D$ really holds.

c) Calculate A^{10} and A^{13} using the diagonalization that you found in part (b).

a) $p(\lambda) = \det(\lambda I - A) = \begin{bmatrix} \lambda+3 & -4 & -6 \\ 2 & \lambda-3 & -6 \\ 0 & 0 & \lambda+1 \end{bmatrix} = (\lambda+1) \begin{vmatrix} \lambda+3 & -4 \\ 2 & \lambda-3 \end{vmatrix}$

$= (\lambda+1)(\lambda^2 - 9 + 8)$

$= (\lambda+1)^2(\lambda-1)$

$\boxed{\lambda_1 = -1}$ $\boxed{\lambda_2 = 1}$

$\boxed{\lambda_1 = -1}$ $(\lambda I - A) = \begin{bmatrix} 2 & -4 & -6 \\ 2 & -4 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\lambda_2 = 1}$ $I - A = \begin{bmatrix} 4 & -4 & -6 \\ 2 & -2 & -6 \\ 0 & 0 & 2 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & -3 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 & 2 & 3 \end{array}\right]$

$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$ $P^{-1}AP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -3 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 4 & 6 \\ -2 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & +3 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D.$

c) $A^{10} = (PDP^{-1})^{10} = P \cdot D^{10} \cdot P^{-1} = P \cdot I \cdot P^{-1} = I$

$A^{13} = (PDP^{-1})^{13} = P \cdot D^{+13} \cdot P^{-1} = P \cdot D P^{-1} = A$.

3. a) In $\mathbb{R}_4$ consider the vectors $\mathbf{v}_1 = (1,-1,2,3)$ and $\mathbf{v}_2 = (1,0,1,2)$. Let V be the subspace of $\mathbb{R}_4$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, and $W = V^{\perp}$ be the orthogonal complement of V in $\mathbb{R}_4$. Find an orthonormal basis for W with respect to the standard inner product of $\mathbb{R}_4$.

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

A basis for W is $\vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

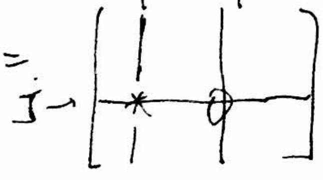
Apply Gram-Schmidt, $\vec{v}_1 = u_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{3}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. $\{\vec{v}_1, \vec{v}_2\}$ orthogonal.

Orthonormal basis $\left\{ \vec{w}_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$.

b) An $n \times n$ matrix A is called a generalized permutation matrix if ~~on~~ in each row and column of A there is only one nonzero entry. Show that the adjoint matrix $Adj(A)$ of a generalized permutation matrix A is also a generalized permutation matrix.

Let $\sigma : \{1,..,n\} \to \{1,..,n\}$ be a function such that $A_{k\ell} = 0$ if $\ell \neq \sigma(k)$. Then, σ is 1-1 and onto, hence a permutation of A. Recall that $Adj(A)_{ij} = c_{ji}$ where $c_{ji} = (-1)^{i+j} \det(M_{ji})$

$M_{ji} =$  = matrix obtained from A by erasing jth row and ith column.

If (j,i) pair is such that, $i \neq \sigma(i)$, then $A_{ji} = 0$. If we erase jth row, then we are erasing a nonzero entry on that row say at column i'. Note that $i' \neq i$. In M_{ji}, then on the i' column all other entries are zero. This gives that $\det(M_{ji}) = 0$. Hence $Adj(A)_{ij} = 0$ when $j \neq \sigma(i)$.

4. For each of the statements below indicate whether the statement is always true or sometimes false. If the statement is true, then justify your answer with a logical argument, and if the statement is false, then give an example to illustrate why the statement is false.

(i) Let V denote the vector space of 2×2 matrices and W be the subset of V formed by all 2×2 matrices such that $A^2 = 2A$. Then W is a subspace of V.

FALSE. The scalar multiplication condition $A \in W \Rightarrow \lambda A \in W$ will fail.

To see that take $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ in W.

$3 \cdot A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ is not in W because $\left. \begin{array}{l} (3A)^2 = 36 \cdot I \\ 2(3A) = 6I \end{array} \right) \neq$.

(ii) Let $S = \{t^2, t+2, 1\}$ and $T = \{t^2 - 1, t, 3\}$ be two different basis for P_2. Then the change of basis matrix $A = P_{S \leftarrow T}$ is lower triangular, i.e. $A_{ij} = 0$ for all $i < j$.

TRUE. $A = P_{S \leftarrow T} = [\,[\vec{w}_1]_S \,\vdots\, [\vec{w}_2]_S \,\vdots\, [\vec{w}_3]_S\,]$.

$\vec{w}_1 = t^2 - 1 = t^2 - 1 \Rightarrow [\vec{w}_1]_S = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$\vec{w}_2 = t = (t+2) - 2 \cdot 1 \Rightarrow [\vec{w}_2]_S = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

$\vec{w}_3 = 3 = 3 \cdot 1 \Rightarrow [\vec{w}_3]_S = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix}$ is lower triangular.

(iii) An $n \times n$ symmetric matrix C is called positive-definite if $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}$ defines an inner product for $\mathbb{R}^n$. If C is positive-definite then the diagonal entries c_{ii} of the matrix C are positive real numbers for all i.

TRUE.

Let $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow$ ith place. Then, $\langle \vec{e}_i, \vec{e}_i \rangle = e_i^T \cdot C \cdot e_i$

$= [0 \cdots 1 \cdots 0] \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ & \ddots & \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix}$

$= c_{ii}$ for all i.

Since $\langle \vec{v}, \vec{w} \rangle$ is an inner product, we must have $\langle \vec{v}, \vec{v} \rangle > 0$ when $\vec{v} \neq 0$. This gives $c_{ii} > 0$ for all i.