Cohomology of Groups - Part 2

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Contents

1 Introduction

These are the lecture notes (Part 2) for the Graduate Course Math 626 Cohomology of Groups, offered in Spring 2024 at Bilkent University. I thank all the students that took the course for taking the course and for the feedbacks and corrections.

These lecture notes are based on some well-known books on Group Cohomology where the author learned the material from. We list some of these books here:

1. Kenneth S. Brown, *Cohomology of Groups*, Springer-Verlag, 1994.
2. A. Adem and R. J. Milgram, *Cohomology of Finite Groups*, Springer-Verlag, 2004.
3. D. Benson, *Representations and Cohomology I, II*, Cambridge University Press, 1998.
4. J. Carlson, et al, *Cohomology Rings of Finite Groups*, Springer, 2003.
5. A. Weibel, *Introduction to Homological Algebra*, Cambridge University Press, 1994.
6. J. J. Rotman, *An Introduction to Homological Algebra*, Second Edition, Springer, 2009.

Conventions: Throughout these notes when we say G is a group we also mean G is a discrete group. The topological groups only appear in Section 3 and there we explicitly say G is a topological group. All R-modules are left R-modules unless it is explicitly stated otherwise. In these every ring R is associative and unital unless otherwise is stated. For the composition of two morphisms f and g, we write fg instead of $f \circ g$.

2 Homology of Groups

2.1 Tensor Products

Let R be an associative unital ring.

Definition 1. Let M be a right R-module and N be a left R-module. Let $F := \mathbb{Z}[M \times N]$ be the free abelian group with basis $M \times N$. Let J be the subgroup of F generated by the elements

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n), \text{ where } m_1, m_2 \in M, n \in N, \\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2), \text{ where } m \in M, n_1, n_2 \in N, \\
&(mr, n) - (m, rn), \text{ where } m \in M, n \in N, r \in R.
\end{aligned}$$

The **tensor product of** M **and** N, denoted by $M \otimes_R N$, is defined to be the quotient group F/J. For every $m \in M$ and $n \in N$, the element $(m, n) + J$ in $M \otimes_R N$ is denoted by $m \otimes n$.

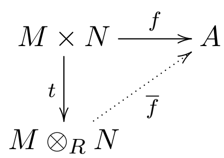
The tensor product $M \otimes_R N$ has a university property that can be explained in terms of R-balanced maps.

Definition 2. Let M be a right R-module, N be a left R-module, and A be an abelian group. A map $f : M \times N \to A$ is **R-balanced** if

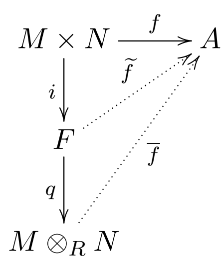
$$
\begin{aligned}
&f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n), \text{ for all } m_1, m_2 \in M, n \in N,\\
&f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2), \text{ for all } m \in M, n_1, n_2 \in N,\\
&f(mr, n) = f(m, rn), \text{ for all } m \in M, n \in N, r \in R.
\end{aligned}
$$

Note that the map $t : M \times N \to M \otimes_R N$ defined by $t(m, n) = m \otimes n$ is an R-balanced map. It is universal in the following sense.

Proposition 3. *Let M be a right R-module and N be a left R-module. For every abelian group A and every R-balanced map $f : M \times N \to A$, there is a unique abelian group homomorphism $\overline{f} : M \otimes_R N \to A$ such that the following diagram commutes:*



Proof. Let $F = \mathbb{Z}[M \times N]$ denote the free $\mathbb{Z}$-module with basis $M \times N$. Let $i : M \times N \to F$ be the inclusion map and $q : F \to F/J = M \otimes_R N$ denote the quotient map. Since F is a free abelian group with basis $M \times N$, the map f uniquely extends an abelian group homomorphism $\widetilde{f} : F \to A$ such that $\widetilde{f} \circ i = f$. This gives a diagram where the small triangle commutes:



By the definition of R-balanced map, the subgroup J in the definition of tensor product is in the kernel of $\widetilde{f}$. By isomorphism theorems, the homomorphism $\widetilde{f}$ induces a unique group homomorphism $\overline{f} : M \otimes_R N \to A$. Since $t = q \circ i$, the homomorphism $\overline{f}$ satisfies $t \circ \overline{f} = f$. $\square$

Remark 4. The tensor product $M \otimes_R N$ can also be defined as the unique abelian group satisfying the universal property with respect to the above extension property with respect to R-balanced maps. Then the tensor product is unique and the existence can be proved using the construction given in Definition 2 and using the argument in Proposition 3.

Lemma 5. *Let M be a S-R-bimodule and N be an R-T-bimodule. Then the tensor product $M \otimes_R N$ is an S-T-bimodule with products defined by $s(m \otimes n)t = sm \otimes nt$ for all $s \in S$, $t \in T$, $m \in M$, and $n \in N$.*

Proof. We need to verify that if $u \in J$, then $sut \in J$. This can be easily checked by verifying it on the generators of J. For example for the last generator we have

$$s[(mr, n) - (m, rn)]t = (smr, nt) - (sm, rnt) \in J.$$

$\square$

Note that the ring R is an R-R-bimodule with left and right multiplication. Then for a right R-module M, the tensor product $M \otimes_R R$ is a right R-module. Similarly if N is a left R-module, then $R \otimes_R N$ is a left R-module.

Lemma 6.

1. *Let M be a right R-module. Then there is an isomorphism of right R-modules*

$$M \otimes_R R \cong M.$$

2. *Let N be a left R-module, then there is an isomorphism of left R-modules*

$$R \otimes_R N \cong N.$$

Proof. (1) Let M be a right R-module. Consider the function $f : M \times R \to M$ defined by $f(m, r) = mr$. The map f is R-balanced, so it extends to an R-module homomorphism $\overline{f} : M \otimes_R R \to M$ defined by $\overline{f}(m \otimes r) = mr$. There is an inverse to $\overline{f}$ defined by $m \to m \otimes 1$. Hence $\overline{f} : M \otimes_R R \to M$ is an isomorphism. It is clear from definition that $\overline{f}$ is an R-module homomorphism (of right R-modules). The proof of (2) can be done in a similar way. □

Proposition 7.

1. *Let $\{M_i\}_{i \in I}$ be a collection of right R-modules and N be a left R-module. Then*

$$\Big(\bigoplus_{i \in I} M_i\Big) \otimes_R N \cong \bigoplus_{i \in I} (M_i \otimes_R N).$$

2. *Let M be a right R-module and $\{N_i\}_{i \in I}$ be a collection of left R-modules. Then*

$$M \otimes_R \Big(\bigoplus_{i \in I} N_i\Big) \cong \bigoplus_{i \in I} (M \otimes_R N_i).$$

Proof. (1) Define

$$f : \Big(\bigoplus_{i \in R} M_i\Big) \times N \to \bigoplus_{i \in I} (M_i \otimes_R N)$$

by $f(\sum_i m_i r_i), n) = \sum_i (m_i r_i \otimes n)$. It is easy to check that f is R-balanced, so it induces an abelian group homomorphism

$$\overline{f} : \Big(\bigoplus_{i \in I} M_i\Big) \otimes_R N \to \bigoplus_{i \in I} (M_i \otimes_R N).$$

The inverse of $\overline{f}$ is defined by $\overline{g}(\sum_j (m_j \otimes n_j)) = \sum_j (i_j(m_j) \otimes n)$. This completes the proof.
(2) This can be proved in a similar way. □

Lemma 8. *Let $f : M \to M'$ be a morphism of right R-modules and $g : N \to N'$ be a morphism of left R-modules. Then there is a unique abelian group homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ such that*

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

Proof. Let $f \times g : M \times N \to M' \otimes_R N'$ be the map defined by $(f \times g)(m,n) = f(m) \otimes g(n)$ for every $m \in M$ and $n \in N$. It is easy to see that $f \times g$ is R-balanced. Then by the universality of the tensor product there is a unique abelian group homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ with desired property. □

Exercise 9. Show that $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \otimes g)$.

Applying tensor product from left or right defines an additive functor which is a right exact functor.

Proposition 10.

1. *Let $0 \to K \xrightarrow{i} M \xrightarrow{\pi} L \to 0$ be a short exact sequence of right R-modules and N be a left R-module, then the sequence*

$$K \otimes_R N \xrightarrow{i \otimes \mathrm{id}_N} M \otimes_R N \xrightarrow{\pi \otimes \mathrm{id}_N} L \otimes_R N \to 0$$

   is exact.

2. *Let M be a right R-module and $0 \to K \xrightarrow{i} N \xrightarrow{\pi} L \to 0$ be a short exact sequence of left R-modules, then the sequence*

$$M \otimes_R K \xrightarrow{\mathrm{id}_M \otimes i} M \otimes_R N \xrightarrow{\mathrm{id}_M \otimes \pi} M \otimes_R L \to 0$$

   is exact.

Proof. (1) It is easy to see that $(\pi \otimes \mathrm{id}_N) \circ (i \otimes \mathrm{id}_N) = (\pi \circ i) \otimes \mathrm{id}_N = 0$. The surjectivity of $\pi \otimes \mathrm{id}_N$ follows from the surjectivity of π. So it remains to show that the extension is exact at $M \otimes_R N$.

Let $u = \sum_j (m_j \otimes n_j)$ be an element of $\ker(\pi \otimes \mathrm{id}_N)$. Then $\sum_j \pi(m_j) \otimes n_j = 0$. This means $\sum_j (\pi(m_j), n_j) \in \mathbb{Z}[L \times N]$ is in the kernel of $q_L : \mathbb{Z}[L \times N] \to L \otimes_R N$. This kernel is generated by elements of the form

$$\begin{aligned} r_1 &= (l_1 + l_2, n) - (l_1, n) - (l_2, n), \text{ where } l_1, l_2 \in L, n \in N, \\ r_2 &= (l, n_1 + n_2) - (l, n_1) - (l, n_2), \text{ where } l \in L, n_1, n_2 \in N, \\ r_3 &= (lr, n) - (l, rn), \text{ where } l \in L, n \in N, r \in R. \end{aligned}$$

Since $\pi : M \to L$ is surjective, for each $l \in M$, we can find $\widehat{l} \in M$ such that $\pi(\widehat{l}) = l$. This means that each of the generators r_i listed above can be lifted to an element $\widehat{r}_i \in \mathbb{Z}[M \times N]$ of the form

$$\begin{aligned} \widehat{r}_1 &= (\widehat{l_1 + l_2}, n) - (\widehat{l}_1, n) - (\widehat{l}_2, n), \text{ where } \widehat{l}_1, \widehat{l}_2, \widehat{l_1 + l_2} \in M, n \in N, \\ \widehat{r}_2 &= (\widehat{l}, n_1 + n_2) - (\widehat{l}, n_1) - (\widehat{l}, n_2), \text{ where } \widehat{l} \in M, n_1, n_2 \in N, \\ \widehat{r}_3 &= (\widehat{lr}, n) - (\widehat{l}, rn), \text{ where } \widehat{lr}, \widehat{l} \in M, n \in N, r \in R \end{aligned}$$

such that $(\pi \times \mathrm{id}_N)(\widehat{r}_i) = r_i$ for $i = 1, 2, 3$. Note that the element $\widehat{r}_2$ is already in $\ker q_M$. Since $\pi(\widehat{l_1 + l_2}) = \pi(\widehat{l}_1 + \widehat{l}_2)$, there is an element $k_{1,2} \in K$ such that $i(k_{i,2}) = \widehat{l_1 + l_2} - \widehat{l}_2 - \widehat{l}_1$. Thus we can write $\widehat{r}_1 = s_1 + (i(k_{12}), n)$ where

$$s_1 = (\widehat{l}_1 + \widehat{l}_2, n) - (\widehat{l}_1, n) - (\widehat{l}_2, n).$$

Since $s_1 \in \ker q_M$, we have $\widehat{r}_1 \in U := \mathrm{im}\,(i \times \mathrm{id}_N) + \ker q_M$. By a similar argument we can show that $\widehat{r}_3 \in U = \mathrm{im}\,(i \times \mathrm{id}_N) + \ker q_M$.

Consider the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}[K \times N] & \xrightarrow{i \times \mathrm{id}_N} & \mathbb{Z}[M \times N] & \xrightarrow{\pi \times \mathrm{id}_N} & \mathbb{Z}[L \times N] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q_L} & & \downarrow{\scriptstyle q_M} & & \downarrow{\scriptstyle q_L} & & \\ 0 & \longrightarrow & K \otimes_R N & \xrightarrow{i \otimes \mathrm{id}_N} & M \otimes_R N & \xrightarrow{\pi \otimes \mathrm{id}_N} & L \otimes_R N & \longrightarrow & 0. \end{array}$$

Note that the top sequence is exact because the kernel of $\pi \times \mathrm{id}_N : \mathbb{Z}[M \times N] \to \mathbb{Z}[L \times N]$ is generated by $(m,n) - (m',n)$ with $\pi(m) = \pi(m')$. Since $\ker \pi = \mathrm{im}\, i$, we obtain that $(m,n) - (m',n) \in \mathrm{im}\,(i \times \mathrm{id}_N)$. Hence the top sequence is exact.

Let $u' = \sum_j (m_j, n_j) \in \mathbb{Z}[M \times N]$ be an element such that $q_M(u') = u$. Since $(\pi \otimes \mathrm{id}_N)(u) = 0$, we have $u'' = (\pi \times \mathrm{id}_N)(u') \in \ker q_L$. We showed above that then u'' can be written as a linear combination of elements of the form r_1, r_2, r_3. Let us denote this linear combination by v. As we showed above that each r_i lifts to an element $\widehat{r}_i$ in $\mathbb{Z}[M \times N]$. Thus the linear combination v will lift to an element $\widehat{v}$ which is a linear combination of elements of the form $\widehat{r}_1, \widehat{r}_2, \widehat{r}_3$. We have $u' - \widehat{v} \in \ker(\pi \times \mathrm{id}_N) = \mathrm{im}\,(i \times \mathrm{id}_N)$. Since by the above argument each $\widehat{r}_i$ lies in the submodule $U = \mathrm{im}\,(i \times \mathrm{id}_N) + \ker q_M$, we obtain that $u' \in U$. We conclude that $u = q_M(u') \in \mathrm{im}\,(i \otimes \mathrm{id}_N)$. This completes the proof of (1). A proof for (2) can be given in a similar way. □

The functors $- \otimes_R N$ and $M \otimes_R -$ are not exact in general. The lack of exactness is source of the left derived functors called the Tor-groups $\mathrm{Tor}^i_R(M, N)$.

Example 11. Consider the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{m_2} \mathbb{Z}/2 \to 0.$$

Applying the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ to this sequence, we obtain a sequence of the form

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/2 \xrightarrow{m_2} \mathbb{Z}/2 \to 0$$

since $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2$ and $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \mathbb{Z}/2$. The map $\mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/2$ is not injective, hence the sequence above is not exact.

Definition 12. A right R-module M is **flat** if the functor $M \otimes_R -$ is exact. A left R-module N is **flat** if the functor $- \otimes_R N$ is exact.

Exercise 13. Prove that a projective module is flat.

Exercise 14. Let p be a prime number and $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z}, (p, b) = 1\}$ denote the ring of p-local integers. Show that $\mathbb{Z}_{(p)}$ is flat over $\mathbb{Z}$.

Another way to prove the right exactness of tensor product functor is to use adjointness of tensor product and Hom-functors. We show this in Section A.2 in Corollary 148.

2.2 Homology of a Chain Complex with Coefficients

In this section we will be using tensor products over a ring R. For the definition and properties of tensor products, we refer the reader to Section 2.1.

Let C_* be a chain complex of right R-modules. Given a left R-module M, we can consider the chain complex $C_* \otimes_R M$ of abelian groups where for each $n \in \mathbb{Z}$, the chain groups are defined by $(C_* \otimes_R M)_n = C_n \otimes_R M$, and for $n \geq 1$ the boundary maps

$$d_n : C_n \otimes_R M \to C_{n-1} \otimes_R M$$

are defined by $d_n(x \otimes m) = d_n(x) \otimes m$ for every $x \in C_n$ and $m \in M$.

Definition 15. Let C_* be a chain complex of right R-modules and M a left R-module. Then the **n-th homology of C_* with coefficients in** M is defined to be the homology group $H_n(C_* \otimes_R M)$.

Example 16. If X is a simplicial complex (or a topological space) and $C_*(X)$ is the simplicial chain complex of X (or the singular chain complex of X), then for every abelian group A, the homology group $H_n(C_*(X) \otimes_{\mathbb{Z}} A)$ is called the **n-th homology group of X with coefficients in** A, and denoted by $H_n(X; A)$.

In some special cases, there is an isomorphism $H_n(C_* \otimes_R M) \cong H_n(C_*) \otimes_R M$. However in general this isomorphism does not hold and it is an interesting question how these two groups are related.

Example 17. Let

$$C_* : 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{m_2} \mathbb{Z}/2 \to 0$$

be the chain complex of abelian groups where $C_1 = C_2 = \mathbb{Z}$ and $C_0 = \mathbb{Z}/2$. Note that C_* is acyclic, i.e. $H_n(C_*) = 0$ for all $n \in \mathbb{Z}$. Take M be the abelian group $\mathbb{Z}/2$. Then the complex $C_* \otimes_{\mathbb{Z}} \mathbb{Z}$ is of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \xrightarrow{\mathrm{id}_{\mathbb{Z}/2}} & \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \end{array}$$

Note that $H_2(C_* \otimes_{\mathbb{Z}} \mathbb{Z}/2) \cong \mathbb{Z}/2$, hence

$$H_2(C_* \otimes_{\mathbb{Z}} \mathbb{Z}/2) \ncong H_2(C_*) \otimes_{\mathbb{Z}} \mathbb{Z}/2.$$

In general, the relation between $H_n(C_*) \otimes_R M$ and $H_n(C_* \otimes_R M)$ can be quite complicated, however when R is a principle ideal domain (PID), the following theorem explains the relationship between these two groups in terms of Tor-groups.

Theorem 18 (Universal Coefficient Theorem for Homology)**.** *Let R be a PID. Suppose that C_* is a chain complex of free R-modules C_* and M is an R-module M. Then for every $n \in \mathbb{Z}$, there is a short exact sequence of R-modules*

$$0 \to H_n(C_*) \otimes_R M \xrightarrow{\varphi} H_n(C_* \otimes_R M) \xrightarrow{\psi} \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

where $\varphi([z] \otimes m) = [z \otimes m]$ for every $[z] \in H_n(C_*)$ and $m \in M$. The sequence is natural with respect to C_* and M, and it splits, but the splitting is not natural.

Here Tor_1^R denotes the Tor-group which is defined in standard textbooks on Homological Algebra. We discuss Theorem 18 in more detail in Section 6.3. Since the ring of integers $\mathbb{Z}$ is a PID, the above theorem holds for abelian groups and it is used for calculating the cohomology groups $H_n(X; A)$ of a space X with coefficients in an abelian group A.

For rings R which are not PID, the relationship between $H_n(C_* \otimes_R M)$ and $H_n(C_*) \otimes_R M$ can not be explained by just one Tor-proup involving only one homology group $H_{n-1}(C_*)$. To illustrate this we given an example below where R is the group ring $\mathbb{Z}C_2$.

Example 19. Let $G = C_2 = \langle g \mid g^2 = 1 \rangle$ be the group with two elements. Consider is a chain complex of $\mathbb{Z}G$-modules:

$$C_* : \cdots \to \mathbb{Z}G \xrightarrow{1+g} \mathbb{Z}G \xrightarrow{1-g} \mathbb{Z}G \xrightarrow{1+g} \mathbb{Z}G \xrightarrow{1-g} \mathbb{Z}G \to 0$$

where all the maps are defined by the multiplication with the element written on the arrow. Recall that if we add the augmentation map $\mathbb{Z}G \to \mathbb{Z}$ to the right hand-side, we get an exact sequence. So, the homology of the complex C_* is given by

$$H_n(C_*) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

The chain complex $C_* \otimes_{\mathbb{Z}G} \mathbb{Z}$ has chain groups $\mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \cong \mathbb{Z}$ for every $n \geq 0$. Calculating the maps induced by ∂_n we obtain that $C_* \otimes_{\mathbb{Z}G} \mathbb{Z}$ is a chain complex of the form

$$\cdots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

The homology groups of this chain complex is given by

$$H_n(C_* \otimes_{\mathbb{Z}G} \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Observe that in this case there is no bound on the dimensions where the homology is nonzero. So a theorem similar to the universal coefficient theorem above does not hold in this case. Tensoring an acyclic chain complex over a group ring $\mathbb{Z}G$ can create homology groups at infinitely many dimensions. In the next section we define the homology of a group G using the homology groups created by tensoring a free resolution F_* with a $\mathbb{Z}G$-module M over $\mathbb{Z}G$.

Exercise 20. Show that the isomorphism $H_n(C_* \otimes_R M) \cong H_n(C_*) \otimes_R M$ holds if M is a free R-module. Conclude that this isomorphism holds for every chain complex of vector spaces over a field $\mathbb{F}$.

Exercise 21. Generalize Example 19 to the group $C_p = \langle g \mid g^p = 1 \rangle$ for any prime p.

2.3 Definition of Group Homology

Let G be a group and k be a commutative ring. Consider the augmentation map $\epsilon : kG \to k$ which sends $\sum_{g \in G} a_g g$ to $\sum_{g \in G} a_g$. The augmentation map ϵ is surjective. Let I_G denote the kernel of ε. We have a short exact sequence of kG-modules

$$0 \to I_G \to kG \xrightarrow{\epsilon} k \to 0.$$

Since ϵ is a ring homomorphism, I_G is an ideal, called the **augmentation ideal** of G.

Lemma 22. *The augmentation ideal I_G is free as a k-module with basis given by the set $Y = \{g - 1 \mid 1 \neq g \in G\}$.*

Proof. Let G_0 denote the set of all nontrivial elements in G. If $u = \sum_{g \in G} a_g g$ is in I_G, then $\sum_{g \in G} a_g = 0$. This gives

$$u = \sum_{g \in G} a_g g = \sum_{g \in G_0} a_g (g - 1)$$

Hence Y spans I_G as a k-module.

Now assume that there exists element $\{a_g\}_{g \in G_0}$ in k such that $\sum_{g \in G_0} a_g (g-1) = 0$. Then

$$\sum_{g \in G_0} a_g g + (\sum_{g \in G_0} g) 1 = 0.$$

Since G is a basis for kG as k-module, we have $a_g = 0$ for all $g \in G_0$. Hence Y is a basis for I_G. □

Definition 23. For a $\mathbb{Z}G$-module M, the **group of coinvariants** is defined by

$$M_G = M / \langle gm - m \mid m \in M, g \in G \rangle.$$

Note that $M_G = M/I_G M$ where I_G is the augmentation ideal. The functor $(-)_G$: kG-Modules $\to$ k-Modules is an additive functor. Note that $(-)_G$ is not an exact functor.

Example 24. Let $G = C_2$ and $k = \mathbb{F}_2$ be the field with two elements. Consider the short exact sequence of kG-modules

$$0 \to k \xrightarrow{\eta} kG \xrightarrow{\epsilon} k \to 0$$

where ϵ is the augmentation map and $\eta(1) = 1 + g$. Applying the functor $(-)_G$ to this short exact sequence, we obtain a sequence of k-modules of the form

$$0 \to k \xrightarrow{0} k \xrightarrow{\mathrm{id}_k} k \to 0.$$

This sequence is not exact. So the functor $(-)_G$ is not an exact functor.

Lack of exactness here is similar to the lack of exactness of taking tensor products. In fact coinvariants of a kG-module can be described in terms of tensor products as follows: Given a kG-module M, we can consider M as a right kG-module via the multiplication $mg = g^{-1}m$ for every $g \in G$ and $m \in M$. Then for kG-modules M and N, we can define the tensor product $M \otimes_{kG} N$ by considering M as a right kG-module. Note that for M and N, the tensor product $M \otimes_k N$ is a kG-module with G-action given by $g(m \otimes n) = (gm \otimes gn)$.

Lemma 25. *For kG-modules M and N, there is an isomorphism*

$$M \otimes_{kG} N \cong (M \otimes_k N)_G$$

where the G-action on $M \otimes_k N$ is defined by $g(m \otimes n) = gn \otimes gn$.

Proof. By definition of tensor products, $M \otimes_{kG} N$ is isomorphic to the quotient k-module $(M \otimes_k N)/L$ where L is the submodule generated by $mg \otimes n - m \otimes gn$ where $m \in M$, $n \in N$, and $g \in G$. Since the right action on M is defined by $mg = g^{-1}m$, the submodule L is generated by $g^{-1}m \otimes n - m \otimes gn = g^{-1}(m \otimes gn) - (m \otimes gn)$. Every element in $M \otimes_k N$ can be uniquely written as $\sum_i m_i \otimes gn_i$, hence the submodule L is equal to the submodule $I_G(M \otimes_k N)$ where I_G is the augmentation ideal and G acts on $M \otimes_k N$ with diagonal action $g(m \otimes n) = gm \otimes gn$. This completes the proof. □

As a direct consequence of the above lemma, we have the following:

Lemma 26. *Let G be a group and k be a commutative ring. For every kG-module M, we have*

$$k \otimes_{kG} M \cong M_G$$

where $M_G := M/I_G M \cong M/\langle gm - m \,|\, g \in G, m \in M\rangle$. Similarly if M is a right $\mathbb{Z}G$-module, then $M \otimes_{kG} k \cong M_G$ where $M_G := M/MI_G$.

Now we are ready to define the homology of a group.

Definition 27. Let G be a group and k be a commutative ring. Let

$$(P_*, \varepsilon) : \cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

be a projective resolution of the trivial module k as a kG-module. For every kG-module M, and for $n \geq 0$, the **n-th homology of G** with coefficients in M is defined by

$$H_n(G; M) := H_n(P_* \otimes_{kG} M).$$

Example 28. (Homology of Cyclic groups) Let n be a positive integer and $G = \langle g \,|\, g^n = 1\rangle$ denote the cyclic group of order n. Consider the free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module

$$(F_*, \varepsilon) \cdots \longrightarrow \mathbb{Z}G \xrightarrow{N_G} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N_G} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

constructed earlier in Part I. Let M be a $\mathbb{Z}G$-module. Then $F_* \otimes_{\mathbb{Z}G} M$ is of the form

$$\cdots \longrightarrow M \xrightarrow{N_G} M \xrightarrow{g-1} M \xrightarrow{N_G} M \xrightarrow{g-1} M \longrightarrow 0\,.$$

From this we conclude that for every $n \geq 0$,

$$H_n(G; M) \cong \begin{cases} M_G & \text{if } n = 0 \\ M^G/N_G M & \text{if } n = odd \\ \ker\{M \xrightarrow{N_G} M\}/(g-1)M & \text{if } n = even > 0 \end{cases}$$

When $M = \mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, we have

$$H_n(G; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Note that for $M = \mathbb{Z}$, the n-homology group $H_n(G;\mathbb{Z})$ the homology group of the chain caomplex

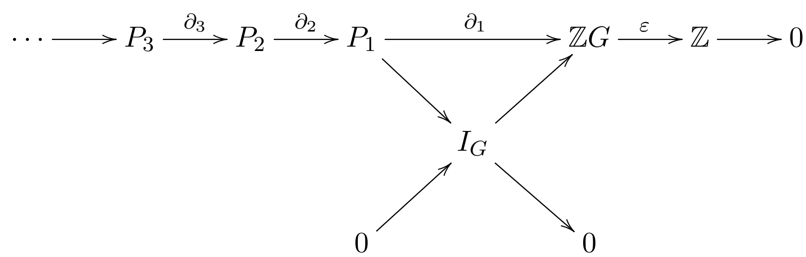
$$(F_*)_G : \cdots \longrightarrow (\mathbb{Z}G)_G \xrightarrow{\times n} (\mathbb{Z}G)_G \xrightarrow{0} (\mathbb{Z}G)_G \xrightarrow{\times n} (\mathbb{Z}G)_G \xrightarrow{0} (\mathbb{Z}G)_G \longrightarrow 0$$

where $(\mathbb{Z}G)_G \cong \mathbb{Z}$.

At low dimensions it is possible to calculate the homology group of G in terms of the structure of the group.

Lemma 29. *Let G be a group and M be a $\mathbb{Z}G$-module. Then $H_0(G;M) \cong M_G$.*

Proof. Recall that for an arbitrary group the projective resolution P_* of a module is constructed by an inductive argument. For the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, we can start the resolution with the augmentation map $\mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$. We take $P_0 = \mathbb{Z}G$. The kernel of the augmentation map is the augmentation ideal I_G. Then P_1 is the projective module such that $P_1 \to I_G$ is surjective. This gives the first two steps of the projective resolution as follows:



Applying the tensor product $- \otimes_{\mathbb{Z}G} M$ to this sequence, we get

$$\cdots \to P_1 \otimes_{\mathbb{Z}G} M \xrightarrow{\partial_1 \otimes \mathrm{id}} M \to 0.$$

Observe that since the tensor product functor is right exact, the map

$$P_1 \otimes_{\mathbb{Z}G} M \to I_G \otimes_{\mathbb{Z}G} M$$

is surjectve. This gives that

$$\mathrm{im}\,(\partial_1 \otimes \mathrm{id}) = \mathrm{im}\,\{I_G \otimes_{\mathbb{Z}G} M \to \mathbb{Z}G \otimes_{\mathbb{Z}G} M\}$$

the image on the right becomes $\mathrm{im}\,\{I_G M \to M\}$ under the isomorphism $\mathbb{Z}G \otimes_{\mathbb{Z}G} M \cong M$. hence, $H_0(G;M) \cong M/I_G M = M_G$. □

3 Topological Definition of Group (Co)homology

3.1 Classifying Space of a Group

Definition 30. Let G be a discrete group. The **classifying space for** G is a connected CW-complex BG which satisfies the condition

$$\pi_n(BG, x_0) \cong \begin{cases} G & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

The universal cover of BG is a contractible space which admits a free action of G. A contractible CW-complex which admits a free action of G is called a **universal space for** G and it is denoted by EG. Note that the orbit space EG/G of EG is the classifying space for G.

Proposition 31. *The universal space EG is unique up to G-homotopy equivalence. The classifying space BG is unique up to homotopy.*

Proof. By equivariant obstruction theory, if X is a free G-CW-complex, then any two G-maps $f, g : X \to EG$ are G-homotopic. If X is another model for a universal space for G, then we can construct maps $f : X \to EG$ and $f' : EG \to X$ such that the compositions $f' \circ f$ and $f \circ f'$ are homotopic to the corresponding identity maps. This gives that EG is unique up to G-homotopy.

If Y is another space satisfying the conditions for the classifying space BG then there is a G-homotopy equivalence $f : \widetilde{Y} \to EG$. This gives an homotopy equivalence $\overline{f} : Y \to EG/G = BG$. □

Remark 32. Note that the classifying space BG is unique only up to homotopy. A space X which has the homotopy type of BG, is called a model for BG.

To prove the existence of classifying spaces for an arbitrary G, there are two different constructions given in the literature. The first one is by Milnor where he constructs the classifying space BG as a realization of a simplicial complex. The second model due to Segal which is defined using simplicial sets (as a special case of the classifying space $B\mathcal{C}$ for a small category $\mathcal{C}$).

Milnor defines BG as the orbit space EG/G where EG is defined as the infinite join

$$EG = G * G * G * \cdots .$$

In the join the group G is considered as a discrete topological space. To describe Milnor's model for EG in more detail, we need to introduce more definitions.

G-simplicial complexes

Definition 33. A simplicial complex X is a G**-simplicial complex** if G acts on the vertex set V of X in such a way that if $\sigma = \{v_0, \ldots, v_n\}$ is a simplex of X, then $g\sigma = \{gv_0, \ldots, gv_n\}$ is also a simplex in X.

The realization $|X|$ of a G-simplicial complex X is a G-space, i.e. a topological space with a continuous G-action $G \times X \to X$. Recall that the realization of a simplicial complex is defined to be the identification space

$$|X| = \Big(\coprod_{n\geq 0}\coprod_{\sigma\in X_n} \{\sigma\}\times \Delta^n\Big)/\sim$$

where $(\sigma, d^i t) \sim (d_i\sigma, t)$ for every $\sigma \in X_n$ and $t \in \Delta^{n-1}$. For every $g \in G$, the action on $|X|$ is defined by $g[(\sigma, t)] = [(g\sigma, t)]$.

Example 34. Let $G = \langle g \,|\, g^2 = 1\rangle \cong C_2$. Consider the 1-dimensional simplicial complex X whose vertex set is $V = \{\pm a, \pm b\}$ and whose 1-simplices are the subsets $\{a, b\}$, $\{a, -b\}$, $\{-a, b\}$, and $\{-a, -b\}$. Let $g \in G$ act on V by $ga = -a$ and $gb = -b$. It is easy to see that with this action, X is a G-simplicial complex. The realization of X is a square with corners given by vertices of X and edges are given by the 1-simplices. Note that $|X|$ is G-homeomorphic to G-space S^1 with the antipodal action. The antipodal action of $G \cong C_2$ on the sphere S^n, $n \geq 0$, is the action defined by $gx = -x$.

An alternative way to define realization of a simplicial complex X is to define it as a subcomplex of a simplex. Suppose that X is a finite simplicial complex and let $|V| = n+1$. Consider the Euclidean space $\mathbb{R}[V] \cong \mathbb{R}^{n+1}$ whose basis is given by the vertex set V. Let Δ^V be the n-simplex defined by

$$\Delta^V = \{t_0v_0 + \cdots + t_nv_n \,|\, t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

Support of a point $t \in \Delta^V$ is the set of vertices $support(t) = \{v_{i_0}, \dots, v_{i_k}\}$ such that in the expression $t = t_0v_0 + \cdots + t_nv_n$ the corresponding coefficients t_{i_j} are nonzero. We define the realization $|X|$ of the simplicial complex X to the subcomplex of Δ^V formed by points $t \in \Delta^V$ whose support is a simplex in X.

Note that this construction easily extends to simplicial complexes with infinitely many vertices. In this case we can take $\mathbb{R}[V]$ to be the infinite dimensional vector space with basis given by V and Δ^V as the subspace consists of formal sums $\sum_i t_iv_i$ in $\mathbb{R}[V]$ with $t_i \geq 0$, $\sum_i t_i = 1$. Note that in the sum there are only finitely many nonzero coefficients t_i which are different than zero. One can define the support of a point in Δ^V a similar way as above and define $|X|$ as the subspace of $R[V]$ formed by points whose support is a simplex in X. Note that the topology on $|X|$ will be induced from the usual metric topology on $R[V]$. Note that these constructions also extend to G-spaces and gives us a G-spaces.

We say that a G-simplicial complex is **admissible** if for every $g \in G$ and for every simplex $\sigma = \{v_0, \dots, v_n\}$ in X, $g\sigma = \sigma$ implies that $gv_i = v_i$ for every $i \in \{0, \dots, n\}$. An admisible G-simplical complex is called **regular** if it satisfies the following condition for every subgroup $H \leq G$: If $h_0, h_1, \dots, h_n \in H$ and $\{v_0, \dots, v_n\}$ and $\{h_0v_0, \dots, h_nv_n\}$ are both simplices of K, then there exists an element h of G such that $hv_i = h_iv_i$ for all i. If a G-simplicial complex X is admissible then for every $H \leq G$, the set of simplices

$$X^H = \{\sigma \in X \,|\, h\sigma = \sigma \text{ for all } h \in H\}$$

is a subcomplex of X. If X is regular, then the set X/G of orbits $[\sigma]_G = \{g\sigma \,|\, g \in G\}$ of simplices in X is a simplicial complex (see [1] for more details).

If X is a simplicial complex, then the set of simplices of X is a partially ordered set under the relation defined by one being a face of the other. Every partially ordered set P has a order complex $\Delta(X)$ defined as the simplicial complex whose faces are the strict chains in P. The face relation is defined by a chain being a subchain of another chain. Given a simplicial complex X the order complex of its poset of simplices is called the subdivision of X and denoted by X. If X is a G-simplicial complex then the subdivision complex $sd(X)$ is admissible, and the second subdivision $sd^2(X)$ is a regular G-simplicial complex for every given G-simplicial complex.

Exercise 35. Prove that if X is a G-simplicial complex, then $sd(X)$ is admissible. Also show that if X is a G-simplicial complex, then $sd^2(X)$ is regular.

Join of simplicial complexes

Definition 36. Let $X = (V_1, S_1)$ and $Y = (V_2, S_2)$ be two simplicial complexes. The join of X and Y is the simplicial complex $X * Y$ whose set of vertices is $V = V_1 \amalg V_2$ and whose set of simplices is given by the set

$$S_1 \cup S_2 \cup \{\sigma \cup \tau \mid \sigma \in S_1, \tau \in S_2\}.$$

Recall that the join of two topological spaces is defined as follows:

Definition 37. Let X and Y be two topological spaces. Then the join of X and Y is the identification space

$$X * Y = (X \times Y \times [0, 1])/\sim$$

where the identifications are given by $(x, y, 0) \sim (x', y, 0)$ and $(x, y, 1) \sim (x, y', 1)$ for all $x, x' \in X$ and $y, y' \in Y$.

We have the following result that links these two join constructions:

Lemma 38. *If X and Y are two simplicial complexes, then there is a homeomorphism*

$$|X * Y| \cong |X| * |Y|.$$

Proof. We write the argument for finite simplicial complexes. It extends easily to infinite simplicial complexes. Let X and Y be finite simplicial complexes with vertex set V and W. Suppose that $|V| = n + 1$ and $|W| = m + 1$. Then X can be realized as a subcomplex of a simplex $\Delta^V \cong \Delta^n$ sitting inside $\mathbb{R}[V]$, and Y can be realized as a subcomplex of $\Delta^W \cong \Delta^m$ sitting inside $\mathbb{R}[W]$. Consider the simplex $\Delta^{V \amalg W} \cong \Delta^{n+m+1}$ sitting inside $R[V \amalg W]$. The realization of the join complex $X * Y$ can be considered as the subcomplex of $\Delta^{V \amalg W}$ consisting of the points

$$t = t_0 v_0 + \cdots + t_n v_n + t'_0 w_0 + \cdots + t'_m w_m$$

whose support is a simplex in $X * Y$, i.e., whose support is either a simplex in X or Y, or it is a union of simplices in X and Y. All such points can be written as

$$t = s \cdot t_X + (1 - s) \cdot t_Y$$

where $s \in [0, 1]$, $t_X \in |X|$, $t_Y \in |Y|$. The set of points of the form $t = s \cdot t_X + (1 - s) \cdot t_Y$ can be identified with points in $|X| * |Y| = |X| \times |Y| \times [0, 1]/\sim$ by senting $t = s \cdot t_X + (1 - s) \cdot t_Y$ to $[(t_X, t_Y, s)]$. This identification gives the desired homeomorphism. □

If X and Y are two G-simplicial complexes, then the join $X * Y$ is a G-simplicial complex. If X and Y are two G-spaces, then $X * Y$ is a G-space with the G-action given by $g[(x, y, t)] = [(gx, gy, t)]$ for $x \in X$, $y \in Y$ and $t \in [0, 1]$. It is easy to see that if X and Y are G-simplicial complexes, then the homeomorphism in Lemma 38 is a G-homeomorphism.

The construction of join defined above can be extended recursively to a join of a finite set of simplices $X_1, \dots, X_n$. In this case one can consider the points of $X_1 \dots \cdots * X_n$ as set of tuples

$$(t_1 x_1, \dots, t_n x_n)$$

where $x_i \in X_i$, $t_i \in [0, 1]$ such that $\sum_i t_i = 1$. If each X_i is a subspace of some $\mathbb{R}^{n_i}$ then we can topologize the join with subspace topology. In general the topology on the join is defined as the finest topology such that the maps

$$t_i : X_1 * \cdots * X_n \to [0, 1] \text{ and } x_i : t_i^{-1}(0, 1] \to X_i$$

are continuous.

If there is a infinite sequence of topological spaces $\{X_i\}_{i \geq 0}$, then we can define $*_{i \geq 0} X_i$ to be the set of infinite sequences

$$(t_1 x_1, \dots, t_n x_n, \cdots)$$

where $x_i \in X_i$, $t_i \in [0, 1]$ such that $\sum_i t_i = 1$. Again if each X_i is a subspace of some $\mathbb{R}^{n_i}$ then we can topologize the join with subspace topology in $\mathbb{R}^\infty$. In general the topology on the join is defined as the finest topology such that the maps

$$t_i : X_1 * \cdots * X_n \to [0, 1] \text{ and } x_i : t_i^{-1}(0, 1] \to X_i$$

are continuous. This topology also coincides with the colimit topology (or weak topology) coming from considering infinite join $*_{i=0}^{\infty} X_i$ as the colimit of finite joins $*_{i=1}^{n} X_i$.

Exercise 39. Verify the statements above about different topologies on joins of spaces.

Classifying Space

If we apply the infinite join construction to the sequence of spaces $\{X_i\}$ where $X_i = G$ considered as a 0-dimensional G-simplicial complex, then we obtain a sequence of G-simplicial complexes $\{\mathcal{E}G_n\}_{n \geq 0}$ where $\mathcal{E}G_0 = G$ and $\mathcal{E}G_n = EG_{n-1} * G$ for every $n \geq 1$. The topological realization of these G-simplicial complexes gives a sequence of G-spaces $\{EG_n\}_{n \geq 0}$ where $EG_n = |\mathcal{E}G_n|$. Note that we have $|EG_0| = G$ and $|EG_n| = |EG_{n-1}| * G$ for all $n \geq 1$. From the definition of join, we see that $\mathcal{E}G_{n-1}$ is a subcomplex of $\mathcal{E}G_n$ for all $n \geq 1$. This gives a directed system

$$EG_0 \xrightarrow{i_0} EG_1 \xrightarrow{i_1} \cdots \to EG_n \xrightarrow{i_n} EG_{n+1} \to \cdots$$

where the maps i_j are the inclusions of subcomplexes. Let EG denote the space defined by

$$EG := \operatorname*{colim}_{n \geq 0} EG_n$$

with the colimit topology, i.e. $U \subseteq EG$ is open if $U \cap EG_n$ is open in EG_n for all $n \geq 0$. Then

Note that EG is the realization of a infinite dimensional simplicial complex

$$\mathcal{E}G = \operatorname*{colim}_{n\geq 0} \mathcal{E}G_n$$

which is the union of all simplicial complexes $\mathcal{E}G_n$. To explain the simplices of $\mathcal{E}G$, we add a point, say p, to the set G and write $G^+ = G \amalg \{p\}$. Then the simplices of $\mathcal{E}G$ are in 1-1 correspondence with the sequences $\sigma = (g_0, g_1, \dots, g_n, \dots)$ such that $g_i \in G^+$ for $i \geq 0$ such that there is at least one $g_i \neq p$, and g_i is different that p for only finitely many g_i's. The dimension of a simplex $\sigma = (g_0, g_1, \dots, g_n, \dots)$ is the number of entries g_i such that $g_i \neq p$. Faces of a simplex are the simplices obtained by replacing one of the g_i's different than p with p.

Proposition 40. *For a discrete group G, the space EG defined above is the universal G-space for G. The orbit space $BG = EG/G$ is a classifying space for G. The space BG is the realization of a simplicial complex.*

Proof. This follows from the fact that taking joins increase connectivity. For $n \geq 1$, we say a topological space X is n-connected if $\pi_i(X, x_0) = 1$ for $i \leq n$. By convention a nonempty set is -1-connected, and a path connected space is 0-connected. By a standard result in algebraic topology, if X is an $(n-1)$-connected space and Y is an $(m-1)$-connected space, then $X * Y$ is $(n+m)$-connected. Since G is -1-connected, $EG_n = G * \cdots * G$ which is the join of $(n+1)$-copies of G is $(n-1)$-connected. This shows that the infinite join EG is weakly contractible, i.e. EG is connected and $\pi_i(EG, e_0) = 1$ for all $i \geq 1$. Since G is a realization of a simplicial complex, it is CW-complex. Then by Whitehead theorem we can conclude that EG is contractible.

Note that EG is the realization of the simplicial complex $\mathcal{E}G$. By taking the barycentric subdivision of $\mathcal{E}G$, we can assume that EG is the realization of a regular G-simplicial complex X. Then

$$BG \cong EG/G \cong |X|/G \cong |X/G|.$$

Hence in Milnor's model BG is the realization of a simplicial complex. □

Example 41. An interesting example to discuss is the case where $G = \langle g \,|\, g^2 = 1 \rangle \cong C_2$. In this case $EG_0 = G$ is the set $\pm v_0$ where v_0 is the unit vector in $\mathbb{R}$. The group G acts on EG_0 by $gv_0 = -v_0$. The (geometric) simplicial complex EG_1 is the square sitting in $\mathbb{R}^2$ with corners $\{\pm v_0, \pm v_1\}$ where v_0 and v_1 are the unit vectors in $\mathbb{R}^2$. It is easy to see that EG_2 will be an octahedron sitting inside $\mathbb{R}^3$ with corners $\{\pm v_0, \pm v_1, \pm v_2\}$. Note that for $n = 0, 1, 2$, EG_n is a G-simplicial complex homeomorphic to S^n with antipodal action. This generalizes to all n easily, so we can show that EG_n is G-homeomorphic to S^n with antipodal action. One can view the universal space EC_2 as the infinite dimensional sphere

$$S^\infty = \operatorname*{colim}_{n\geq 0} S^n$$

with antipodal action. The classifying space BC_2 is the orbit space $S^\infty/\sim$ which is the infinite dimensional real projective space

$$\mathbb{R}P^\infty = \operatorname*{colim}_{n\geq 0} \mathbb{R}P^n.$$

For a specific G, there could be models for EG simpler than the model described above.

Example 42. Let $G = \mathbb{Z}$ be the group of integers. Then we can take the space of real numbers $\mathbb{R}$ as a model for the universal space EG. Since $G = \mathbb{Z}$ acts freely on $\mathbb{R}$ by translations and since $\mathbb{R}$ is contractible, $EG = \mathbb{R}$ is universal space for $G = \mathbb{Z}$. Then the classifying space BG for G is $EG/G = \mathbb{R}/\mathbb{Z} \cong S^1$. Note that in this case the classifying space BG is finite dimensional. We will show later that when G has a torsion element (i.e. has a cyclic subgroup with finite order), then BG can be finite dimensional.

Another example of a specific model for EG is the following:

Example 43. Let p be an odd prime and $G = \langle g \mid g^p = 1 \rangle \cong C_p$ be the cyclic group of order p. Then G acts freely on a circle $X = S^1$ defined by rotation with $2\pi/p$ degrees. Then taking the join of X with itself infinitely many times, we obtain an infinite dimensional G-space $EG = *_{i=1}^{\infty} S^1$ where G-acts freely on EG and EG is contractible. So EG is a universal space for $G = C_p$.

Note that in this case $EG_0 = S^1$, $EG_1 = S^1 * S^1 \cong S^3$, and $EG_n = S^{2n+1}$ for $n \geq 0$. For each $n \geq 0$, to describe the free $G = C_p$ action on $EG_n = S^{2n+1}$ we consider $X = S^{2n+1}$ as a subspace of $\mathbb{C}^n$, as the n-tuple of complex numbers $(z_1, \ldots, z_n)$ satisfying $|z_1| + \cdots + |z_n| = 1$. The $g \in G$ action on X is defined by

$$g \cdot (z_1, z_2, \ldots, z_n) = (e^{2\pi i/p} z_1, e^{2\pi i/p} z_2, \ldots, e^{2\pi i/p} z_n).$$

We conclude this section with an observation on classifying spaces.

Lemma 44. *Let G and H be two groups. Then $EG \times EH$ is a universal space for $G \times H$ with a free $G \times H$-action defined by $(g, h)(e_1, e_2) = (ge_1, he_2)$. For classifying spaces, we have*

$$B(G \times H) \cong BG \times BH.$$

Proof. It is clear that the diagonal $G \times H$-action on $EG \times EH$ is free and $EG \times EH$ is contractible. So we have a $G \times H$-homotopy equivalence $E(G \times H) \to EG \times EH$. Taking orbits this gives a homotopy equivalence

$$B(G \times H) = E(G \times H)/(G \times H) \cong EG/G \times EH/H \cong BG \times BH.$$

□

3.2 Topological Definition of Group Cohomology

Let G be a discrete group and k be a commutative ring. For a simplicial complex X the simplicial chain complex of X with coefficients in k is a chain complex $C_*(X; k)$ with the n-chain module $C_n(X; k)$ defined as free k-module whose basis is given by n-dimensional simplices in X. The n-th cohomology group of X with coefficients in a k-module A is defined to be the cohomology of the cochain complex $\mathrm{Hom}_k(C_*(X); A)$.

If X is a G-simplicial complex, G-permutes the simplices in X. Since the basis for $C_n(X; k)$ is permutes by G, we can consider $C_n(X; k)$ as a permutation kG-module with the G-action defined by

$$g\Big(\sum_{\sigma \in X_n} \lambda_\sigma \sigma\Big) = \sum_{\sigma \in X_n} \lambda_\sigma (g\sigma).$$

Note that this action commutes with the boundary maps, so $C_*(X;k)$ is a chain complex of permutation kG-modules. Moreover if the G-action on X is free, then the chain complex $C_*(X;k)$ is a chain complex of free kG-modules.

Let A be a k-module. If we calculate the homology of the chain complex $C_*(X;k)\otimes_k A$, we obtain homology groups $H_n(X;A)$ which are kG-modules. Similarly, the cohomology groups $H^n(X;A) = H^n(\mathrm{Hom}_k(C_*(X;k),A))$ are also kG-modules with the G-action induced by the G-action on $C_*(X;k)$.

We have another (co)homology definition for X when the coefficients are actually kG-modules.

Definition 45. Let X be a G-simplicial complex and M be a kG-module.

1. The G-homology of X with coefficient in M is the homology of the chain complex

$$C_*(X;k)\otimes_{kG} M.$$

2. The cohomology of X with coefficient in M is the cohomology of the cochain complex

$$\mathrm{Hom}_{kG}(C_*(X;k),M).$$

Unfortunately there is no agreed terminology and notation for these (co)homology groups. In some papers these (co)homology groups are denoted by $H_n^G(X;M)$ and $H_G^n(X;M)$. We will use this notation too. Note that we have the following:

Lemma 46. *Let k denote the trivial kG-module. Then we have isomorphisms*

$$H_n^G(X;k)\cong H_n(X/G;k) \quad \textit{and} \quad H_G^n(X;k)\cong H^n(X/G;k).$$

Proof. For every G-simplicial complex X, we have

$$C_*(X;k)\otimes_{kG} k\cong C_*(X/G;k).$$

Similarly we have $\mathrm{Hom}_{kG}(C^*(X;k),k)\cong \mathrm{Hom}_k(C_*(X/G;k),k) = C^*(X/G;k)$. □

The (co)homology group definitions above extends to the singular homology in a similar way. If X is an arbitrary topological space, then its universal cover $\widetilde{X}$ has a free G-action where $G=\pi_1(X,x_0)$ is the fundamental group of X. the orbit space of the G-action on $\widetilde{X}$ is homeomorphic to X.

Definition 47. Let X be a topological space and $G=\pi_1(X,x_0)$. Then for every kG-module M,

1. the homology of X with local coefficients in M is defined by

$$H_n^{loc}(X;M) := H_n(C_*(\widetilde{X};k)\otimes_{kG} M),$$

2. the cohomology of X with local coefficients M is defined by

$$H^n_{loc}(X;M) := H^n(\mathrm{Hom}_{kG}(C_*(\widetilde{X};k),M)).$$

Note that when M is the trivial kG-module k, then the local (co)homology coincides with the ordinary (co)homology since in this case

$$H^n_{loc}(X;k) := H^n(\text{Hom}_{kG}(C_*(\widetilde{X};k),k)) \cong H^n(C^*(X;k)).$$

Similar argument holds for homology.

Now we are ready to give the topological definition of group (co)homology.

Definition 48. (Topological definition of group (co)homology) Let G be a discrete group and M be a kG-module. Then,

1. The homology of G with coefficients in M is defined by
$$H_n(G;M) := H^{loc}_n(BG;M) \cong H^n(C_*(EG;k) \otimes_{kG} M),$$
2. The cohomology of G with coefficients is M is defined by
$$H^n(G;M) := H^n_{loc}(BG;M) \cong H^n(\text{Hom}_{kG}(C_*(EG;k),M))$$

Proposition 49. *The topological definition of group (co)homology coincides with the algebraic definition.*

Proof. The topological definition of group cohomology is given by

$$H^*(G;M) \cong H^n(\text{Hom}_{kG}(C_*(EG;k),k).$$

Since G acts freely on EG, the chain complex $C_*(EG;k)$ is a chain complex of free kG-modules. Since EG is contractible, the augmented complex

$$\widetilde{C}_*(EG;k) : \cdots \to C_n(EG;k) \xrightarrow{\partial_n} C_{n-1}(EG;k) \to \cdots \to C_1(EG,k) \xrightarrow{\partial_1} C_0(EG,k) \xrightarrow{\epsilon} k \to 0$$

is exact. Hence this gives a free resolution of k as a kG-module. Since the group cohomolgoy was defined as the cohomology of the cochain complex $\text{Hom}_{kG}(F_*,M)$ for some free resolution $F_* \to k$. We also showed that the definition is independent from the chosen free resolution. Hence we can conclude that topological and algebraic definitions coincide. The argument for homology is similar. □

Note that if M is the trivial kG-module k, then by the arguments above, we have

$$H_n(G;k) \cong H_n(BG;k) \quad \text{and} \quad H^n(G;k) \cong H^n(BG;k).$$

This says that (co)homology of a group G is the (co)homology of its classifying space BG.

3.3 Classifying Space of a Small Category

Another approach to constructing models for classifying space BG for a group G is using simplicial sets and construct BG it as a special case of the classifying space $B\mathcal{C}$ of a small category $\mathcal{C}$. We now describe this approach.

The **simplex category** Δ is the category whose objects are the ordered finite sets $[n] = \{0,1,\ldots,n\}$ for all $n \geq 0$. Morphisms in Δ are given by order preserving functions $\varphi : [m] \to [n]$, i.e., function that satisfy that if $x \leq y$ then $\varphi(x) \leq \varphi(y)$. The opposite category of Δ is denoted by Δ^{op}.

Definition 50. A **simplicial set** X is a functor $X : \Delta^{op} \to Sets$. For each $n \geq 0$, we denote $X([n])$ by X_n and call it the set of n-simplices in X. For each $\varphi : [m] \to [n]$, we denote the induced map $X(\varphi) : X([n]) \to X([m])$ by $\varphi^* : X_n \to X_m$. A morphism $f : X \to Y$ between two simplicial complexes is a natural transformation between the functors $X, Y : \Delta^{op} \to Sets$.

One of the important observations on the simplex category is that the morphisms in Δ are generated by certain type of morphisms called face and degeneracy maps. For each $0 \leq i \leq n$, let $d^i : [n-1] \to [n]$ be the injective monotone map that skips the element $i \in [n]$. The induced map $d_i := (d^i)^* : X_n \to X_{n-1}$ is called the i-th **face map** of X. For each $0 \leq i \leq n$, let $s^i : [n+1] \to [n]$ be the monotone surjective function that sends i and $i+1$ to i. The induced map $s_i := (s^i)^* : X_n \to X_{n+1}$ is called the i-th **degeneracy map** of X.

To define the realization of a simplicial set, we use geometric n-simplex

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0, \sum t_i = 1\} \subseteq \mathbb{R}^{n+1}.$$

Note that in this section we denote the geometric n-simplex by $|\Delta^n|$ instead of Δ^n. In simplicial set theory Δ^n denotes the simplicial set with k-simplices given by $\mathrm{Mor}_\Delta([k], [n])$.

We consider the points of $|\Delta^n|$ as linear conbinations $\sum_i t_i e_i$ where e_i is the i-th unit vector in $\mathbb{R}^{n+1}$. For each morphism $\varphi : [m] \to [n]$ in Δ, we define $\varphi_* : |\Delta^m| \to |\Delta^n|$ to be the map defined by

$$\varphi_*\Big(\sum_{i=0}^{m} t_i e_i\Big) = \sum_{i=0}^{m} t_i e_{\varphi(i)}.$$

Definition 51. Let X be a simplicial set. The **geometric realization of** X is defined as the identification space

$$|X| = \Big(\coprod_{n\geq 0} \coprod_{\sigma \in X_n} \{\sigma\} \times |\Delta^n|\Big) / \sim$$

where $(\varphi^*(\sigma), t) \sim (\sigma, \varphi_*(t))$ for every morphism $\varphi : [m] \to [n]$ in Δ.

Given a small category $\mathcal{C}$, the nerve $\mathcal{NC}$ of $\mathcal{C}$ is defined as the simplicial set such that $\mathcal{NC}_n$ is the set of all chains

$$\sigma : c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} c_2 \to \cdots \xrightarrow{\alpha_n} c_n$$

of composable morphisms in $\mathcal{C}$. We can consider each such chain as a functor $\sigma : [n] \to \mathcal{C}$. Then for each morphism $\varphi : [m] \to [n]$, we define the induced map $\varphi^*(\sigma)$ as the composition

$$\sigma \circ \varphi : [m] \xrightarrow{\varphi} [n] \xrightarrow{\sigma} \mathcal{C}.$$

In particular, for $0 \leq i \leq n$, we have

$$d_i(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n) = \begin{cases} c_1 \xrightarrow{\alpha_2} c_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} c_n & \text{if } i = 0 \\ c_0 \xrightarrow{\alpha_1} \cdots \to c_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} c_{i+1} \to \cdots \xrightarrow{\alpha_n} c_n & \text{if } 0 < i < n \\ c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} c_{n-1} & \text{if } i = n. \end{cases}$$

For $i \in \{0, \ldots, n\}$, the degeneracy map $s_i : \mathcal{NC}_n \to \mathcal{NC}_{n+1}$ is defined by

$$s_i(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n) = (c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} c_n).$$

It is a standard result in simplicial set theory that the realization $|X|$ of the simplicial complex X is a CW-complex where cells of $|X|$ are in 1-1 correspondence with the nondegenerate simplices in X (a simplex X is nondegenerate if it is not in the image of the degeneracy map s_i for some i).

Definition 52. The classifying space $B\mathcal{C}$ of a small category $\mathcal{C}$ is defined to be the realization of the nerve $\mathcal{NC}$ of $\mathcal{C}$.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between two small categories, then it induces a simplicial map $\mathcal{NC} \to \mathcal{ND}$ which gives a continuous map $BF : B\mathcal{C} \to B\mathcal{D}$. A natural transformation $\eta : F \Rightarrow F'$ between to functors $F, F' : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathcal{C} \times \{0 \to 1\}\mathcal{D}$ where $\{0 < 1\}$ is the poset category with two objects 0 and 1 satisfying the order relation $0 < 1$. This gives a continuous map

$$H : B\mathcal{C} \times [0, 1] \to B\mathcal{D}$$

such that $H(x, 0) = BF(x)$ anad $H(x, 1) = BF'(x)$. This defines a homotopy between BF and BF'.

Lemma 53. *An object $c_0 \in \mathrm{Ob}\,\mathcal{C}$ is called an initial object if for every $c \in \mathrm{Ob}\,\mathcal{C}$ there is a unique morphism $f : c_0 \to c$ in $\mathcal{C}$. If $\mathcal{C}$ has an initial object then $B\mathcal{C}$ is a contractible space.*

Proof. Let $\mathcal{C}_0$ denote the subcategory of $\mathcal{C}$ with one object c_0 and with only one morphism which is the identity morphism of c_0. Let $const_{c_0} : \mathcal{C} \to \mathcal{C}_0$ denote the functor that takes very object in $\mathcal{C}$ to c_0 and every morphism to the identity morphism of c_0. Let $inc_{c_0} : \mathcal{C}_0 \to \mathcal{C}$ denote the functor defined by the inclusion map. It is clear that the composition $const_{c_0} \circ inc_{c_0}$ is the identity functor of the category $\mathcal{C}_0$. The composition $F := inc_{c_0} \circ const_{c_0}$ is not equal to the identity functor, however there is a natural transformation $\eta : F \Rightarrow \mathrm{id}_{\mathcal{C}}$ defined by $\eta_c = \alpha : c_0 \to c$ where $\alpha : c_0 \to c$ is the unique morphism in $\mathcal{C}$ from the initial element c_0 to c. We conclude that the maps $\mathrm{id}_{B\mathcal{C}} : B\mathcal{C} \to B\mathcal{C}$ and $F : B\mathcal{C} \to B\mathcal{C}$ are homotopic. Since F is the constant map, we conclude that $B\mathcal{C}$ is contractible. □

Example 54. A poset P can be considered as a category whose objects are the elements of P with a single morphism $x \to y$ if $x \leq y$. Then the nerve of P as a category is the simplicial complex $\mathcal{N}P$ whose n-simplices are given by chains $\sigma = (x_0 \leq x_1 \leq \cdots \leq x_n)$ of elements in P. The degeneracy map is given by doubling an element, so nondegenerate simplices of $\mathcal{N}P$ are the strict chains of elements of P. The classifying space BP is homeomorphic to the order complex $\Delta(P)$. If P has a minimum, i.e., an element $x_0 \in P$ such that $x_0 \leq x$ for all $x \in P$, then P has an initial element as a category, so BP is contractible in this case.

Let G be a group. Consider the category with one object $*$ and with morphisms given by group elements $g \in G$. The composition of two morphisms $g, h \in G$ is defined by group multiplication gh. We denote this category by $\mathbf{G}$. Note that the nerve $\mathcal{N}\mathbf{G}$ is the simplicial complex whose n-simplices are given by n-tuples $(g_1, \dots, g_n)$ to denote the chain of maps $* \xleftarrow{g_1} * \xleftarrow{g_2} * \cdots * \xleftarrow{g_n} *$ in the category $B\mathbf{G}$. Then the boundary maps $d_i : B\mathbf{G}_n \to B\mathbf{G}_{n-1}$ are given by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, g_3, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

Note the similarity between this formula and the formula for the boundary maps in the bar notation. We explain this relation later in the section.

For $i \in \{0, \dots, n\}$, the degeneracy map $s_i : B\mathbf{G}_n \to B\mathbf{G}_{n+1}$ is given by

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n).$$

The classifying space $B\mathbf{G}$ is a CW-complex whose n-cells are in 1-1 correspondence with n-tuples $(g_1, \dots, g_n)$ where $g_i \neq 1$ for all i.

We will show below that the classifying space of this category is a model for BG and its chain complex gives the normalized bar complex. To prove this we need to introduce another category.

Definition 55. Let G be a group. Consider the category $\mathcal{E}_G$ whose objects are the elements $g \in G$ and for every $g, h \in G$, there is a single morphism $g \to h$.

The category $\mathcal{E}_G$ is an example of a G-category. A category $\mathcal{C}$ is called a G**-category** if there is an action of G on the set of objects and on the set of morphisms such that if for every morphism $\varphi : x \to y$ in $\mathcal{C}$, $g\varphi$ is a morphism $gx \to gy$ in $\mathcal{C}$. A G-category $\mathcal{C}$ can be considered as a functor $F_{\mathcal{C}} : \mathbf{G} \to Cat$ where $F_{\mathcal{C}}(*) = \mathcal{C}$. Here Cat denotes the category of small categories. Note that if $\mathcal{C}$ is a G-category then $\mathcal{NC}$ is a G-simplicial set. A G**-simplicial set** is defined as a simplicial set X with a G-action on its set of simplices X_n for $n \geq 0$ such that the G-action of simplices commutes with the boundary and degeneracy maps. If $\mathcal{C}$ is a G-category, then the realization $B\mathcal{C} = |\mathcal{NC}|$ is a G-CW-complex.

Lemma 56. *The classifying space $B\mathcal{E}_G = |\mathcal{N}\mathcal{E}_G|$ is a model for the universal G-space EG for G.*

Proof. $B\mathcal{E}_G$ is a G-CW-complex whose cells are in 1-1 correspondence with the nondegenerate simplices in $X = \mathcal{N}\mathcal{E}_G$. A simplex $\sigma = (g_0 \to g_1 \to \cdots \to g_n)$ in X_n is nondegenerate if $g_i \neq g_{i+1}$ for all i. The G-action on σ is defined by $g\sigma = (gg_0 \to gg_1 \to \cdots \to gg_n)$. It is clear that $g\sigma = \sigma$ only when $g = 1$. So the G-action on $\mathcal{N}\mathcal{E}_G$ is free, hence $B\mathcal{E}_G$ is a free G-CW-complex. To see that $B\mathcal{E}_G$ is contractible, observe that in $\mathcal{E}_G$ every object is an initial object (also terminal) since there is a unique morphism between any two objects. Hence $B\mathcal{E}_G$ is a model for the universal G-space for G. □

For a G-simplicial set X, the orbit simplicial complex X/G is the simplicial complex whose n-simplices are given by the G-orbits of n-simplices of X. This the boundary and degeneracy maps commutes with the G-action, they induce boundary and degeneracy maps for the orbit simplicial set X/G.

Proposition 57. *There is a simplicial isomorphism between $(\mathcal{N}\mathcal{E}_G)/G$ and $\mathcal{N}\mathbf{G}$. Therefore, $B\mathcal{E}_G/G$ is homeomorphic to $B\mathbf{G}$, and $B\mathbf{G}$ is a model for BG.*

Proof. Consider the simplicial map $\xi : \mathcal{N}\mathcal{E}_G \to \mathcal{N}\mathbf{G}$ that sends $g_0 \to g_1 \to \cdots \to g_n$ to the tuple $(g_0^{-1}g_1, g_1^{-1}g_2, \cdots, g_{n-1}^{-1}g_n)$. Note that this map if G-equivariant because

$$\begin{aligned}\xi(g(g_0, g_1, \dots, g_n)) &= \xi(gg_0, gg_1, \dots, gg_n) = ((gg_0)^{-1}gg_1, \dots, (gg_{n-1})^{-1}gg_n) \\ &= (g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n) = \xi(g_0, \cdots, g_n)\end{aligned}$$

Hence ξ induces a simplicial map $\overline{\xi} : (\mathcal{N}\mathcal{E}_G)/G \to \mathcal{N}\mathbf{G}$. The inverse map to $\overline{\xi}$ is given by $(g_1, g_2, \dots, g_n) \to (1 \to g_1 \to g_1g_2 \to \cdots \to g_1 \cdots g_n)$. It is clear that these simplicial maps are inverse to each other and $\overline{\xi}$ is a simplicial isomorphism.

If X is a G-simplicial set, then $|X/G|$ is homeomorphic to the orbit space $|X|/G$. This gives that there is a homeomorphism $B\mathcal{E}_G/G = |\mathcal{N}\mathcal{E}_G|/G \cong |(\mathcal{N}\mathcal{E}_G)/G|$. By the simplicial isomorphism we just proved, we obtain $B\mathcal{E}_G/G \cong |\mathcal{N}\mathbf{G}| = B\mathbf{G}$. Since $B\mathcal{E}_G$ is a model for universal G-space, we conclude that $B\mathbf{G}$ is a model for classifying space $b\mathbf{G}$. □

Cellular chain complex of CW-complex X is a chain complex $C_*(X; k)$ whose n-chain module $C_n(X; k)$ is the free k-module whose basis is the set of n-dimensional cells in X. The boundary maps are defined by using attaching maps. for more details we refer the reader to ?. The cellular cohomology of X with coefficients in a k-module A is defined by the cohomology of the cochain complex $\mathrm{Hom}_k(C_*(X; k), A)$. If X is a G-CW-complex, then the cellular chain complex $C_*(X; k)$ is a chain complex of kG-modules. For a kG-module M, we defined the cohomology groups $H^n_G(X; M)$ to the cohomology groups of the cochain complex

$$C^*_G(X; M) := \mathrm{Hom}_{kG}(C_*(X; k), M).$$

We proved above that $B\mathcal{E}_G$ is a free G-CW-complex which is contractible. Then the augmented cellular chain complex $C_*(B\mathcal{E}_G; k) \to k$ gives a free resolution of k as a kG-module. Note that the k-basis for $C_n(B\mathcal{E}_G; k)$ is given by the set of all chains $(g_0 \to g_1 \to \cdots \to g_n)$ such that $g_i \neq g_{i+1}$ for all i. The boundary map d_i is given by

$$d_i(g_0 \to g_1 \to \cdots \to g_n) = (g_0 \to \cdots \to \widehat{g_i} \to \cdots \to g_n).$$

This shows that the free resolution $C_*(B\mathcal{E}_G; k) \to k$ is the normalized standard resolution.

The CW-complex $B\mathbf{G}$ has a cellular chain complex $C_*(B\mathbf{G}; k)$ where the n-chain module $C_n(B\mathbf{G}; k)$ is a free k-module with basis given by tuples $(g_1, \dots, g_n)$. Since the boundary maps d_i are defined in s similar way to the boundary maps in the bar resolution we see that the chain complex gives the chain complex for bar resolution for homology. The cohomology version can also be seen as the cochain complex with coefficients in a local coefficient systems.

Example 58. Let $G = \langle g \,|\, g^2 = 1\rangle$. Then the nondegenerate n-simplices of $\mathcal{N}\mathcal{E}_G$ are chains of morphisms $g_0 \to g_1 \to \cdots \to g_n$ where $g_i \neq g_{i-1}$ for all i. There are only two chain of this form $\sigma = (1 \to g \to 1 \to \cdots \to g_n)$ and $g\sigma = (g \to 1 \to g \to \cdots \to gg_n)$ where $g_n = 1$ or g depending on whether n is odd or even. For $n = 0$, where have $\sigma^0 = (1)$ and $g\sigma^0 = (g)$. For $n = 1$, we have $\sigma^1 = (1, g)$ and $g\sigma^1 = (g, 1)$. Note that $d^0(\sigma^1) = g\sigma^0$ and $d^1(sigma^1) = \sigma^0$. From this we see that the one skeleton of $B\mathcal{E}_G$ is a circle S^1 obtained by two 1-cells to two 0-cells around the boundaries so that the G-action swaps these 0-cells and 1-cels. In the 2-skeleton we will have two new 2-discs will be attached to this circle, again G will act on these two new cells by swapping them. It is not hard to see that 2-skeleton will be a 2-sphere S^2 with antipodal action. In general the n-skeleton will be the n-sphere S^n with antipodal action. Then $B\mathcal{E}_G$ is the infinite sphere $S^\infty = \mathrm{colim}_{n\geq 0} S^n$ with antipodal action. Note that $B\mathcal{E}_G$ is not a G-simplicial complex it is G-CW-complex.

The nondegenerate simplices of $B\mathbf{G}$ are the sequences of $(g_0, \dots, g_n)$ where non of the entries are equal to one. Then there is only one simplex at each dimension $\tau^n = (g, g, \dots, g)$. Note that $d^i(\tau^n)$ is degenerate if $0 < i < n$ and $d^i(\tau^n) = \tau^{n-1}$ when $i = 0, n$. The 0-skeleton will be a point $\{x_0\}$, the 1-skeleton will be a circle S^1 obtained by attaching a 1-simplex by gluing both ends to $\{x_0\}$. The 2-skeleton is be obtained by attaching a 2-simplex by gluing $d_0(0,1,2) = (1,2)$ and $d_2(0,1,2) = (0,1)$ to the 1-skeleton S^1 and $d_1(0,1,2) = (0,2)$ to the point $\{x_0\}$. This will give an $\mathbb{R}P^2$. Continuing this way we see that the n-skeleton of $B\mathbf{G}$ is homeomorphic to $\mathbb{R}P^n$ and $B\mathbf{G}$ is $\mathbb{R}P^\infty$.

4 Restriction, Transfer, and Cartan-Eilenberg Theorem

4.1 Restriction and Transfer on the Cochain Level

Throughout this section let G be a group and H be a subgroup of G. As before, k denotes a commutative ring and kH and kG denote the corresponding group rings.

We can view kH as a subring of kG and consider a kG-module M as a kH-module via restriction. We denote this kH-module by $\mathrm{Res}_H^G M$. Another common notation for restriction is $M \downarrow_H^G$ which we will not be using.

For $M = kG$, we have the following isomorphism of kH-modules:

$$\mathrm{Res}_H^G(kG) \cong \bigoplus_{gH \in G/H} kH.$$

This gives in particular that $\mathrm{Res}_H^G(kG)$ is a free kH-module. As a consequence we observe the following:

Lemma 59. *If $F_* \to k$ is a free kG-resolution of k, then $\mathrm{Res}_H^G F_* \to k$ is a free kH-resolution of k.*

Proof. Every free kG-module F is isomorphic to $\oplus_{i \in I} kG$ for some indexing set I, hence $\mathrm{Res}_H^G F$ is isomorphic to $\oplus_{j \in J} kH$. Thus for every free kG-resolution $F_* \to k$, $\mathrm{Res}_H^G F_n$ is a free kH-module for every $n \geq 0$. Since the restriction functor does not change the homology of a chain complex, the complex $\mathrm{Res}_H^G F_* \to k$ is a free resolution of k. □

Given a kG-module homomorphism $f : M' \to M$, we can view it as a kH-module homomorphism $\mathrm{Res}_H^G f : \mathrm{Res}_H^G M' \to \mathrm{Res}_H^G M$ by restricting the kG-module structure on M' and M to kH. This gives a k-module homomorphism

$$\mathrm{Res}_H^G : \mathrm{Hom}_{kG}(M', M) \to \mathrm{Hom}_{kH}(\mathrm{Res}_H^G M', \mathrm{Res}_H^G M).$$

Proposition 60. *If $F_* \to k$ is a free kG-resolution of k, and M is a kG-module, then there is a chain map*

$$\mathrm{Res}_H^G : \mathrm{Hom}_{kG}(F_*, M) \to \mathrm{Hom}_{kH}(\mathrm{Res}_H^G F_*, \mathrm{Res}_H^G M).$$

The induced homomorphism

$$\mathrm{Res}_H^G : H^n(G; M) \to H^n(H, \mathrm{Res}_H^G M)$$

is called the **restriction map** *for group cohomology.*

Proof. This follows from Proposition 59. □

Now we will define a map in the other direction called transfer map. For this we need to assume that the index $|G : H|$ is finite. As before we first define the transfer homomorphism on the chain complex level.

Let M' and M be kG-modules, and $f : \mathrm{Res}^G_H M' \to \mathrm{Res}^G_H M$ be a kH-module homomorphism. Let us fix a set of coset representatives $G/H = \{g_1H, \dots, g_kH\}$ where $k = |G : H|$. Consider the k-module homomorphism $\mathrm{Tr}^G_H f : M' \to M$ defined by

$$(\mathrm{Tr}^G_H f)(m) = \sum_{i=1}^{k} g_i f(g_i^{-1} m)$$

for every $m \in M'$.

Lemma 61. *The map $\mathrm{Tr}^G_H f : M' \to M$ does not depend on the coset representatives that are chosen, and it defines a kG-module homomorphism.*

Proof. Let $G/H = \{g'_1H, \dots, g'_kH\}$ be another set of coset representatives. Then for each $i \in \{1, \dots, k\}$, there is an element $h_i \in H$ such that $g'_i = g_i h_i$. We have

$$\sum_{i=1}^{k} g'_i f(g_i'^{-1} m) = \sum_{i=1}^{k} g_i h_i f(h_i^{-1} g_i^{-1} m) = \sum_{i=1}^{k} g_i f(g_i^{-1} m).$$

This shows that transfer map $\mathrm{Tr}^G_H f$ does not depend on the coset representatives that are chosen.

For every $g \in G$ and $m \in M'$, we have

$$(\mathrm{Tr}^G_H f)(gm) = \sum_{i=1}^{k} g_i f(g_i^{-1} gm) = g\Big(\sum_{i=1}^{k} g^{-1} g_i f(g_i^{-1} gm)\Big).$$

Since the set $\{g^{-1}g_1H, \dots, g^{-1}g_kH\}$ is another set of coset representatives, the sum in the last expression is equal to $\mathrm{Tr}^G_H f$. This shows that $\mathrm{Tr}^G_H f$ is a kG-module homomorphism. $\square$

The map that sends each kH-homomorphism $f : \mathrm{Res}^G_H M' \to \mathrm{Res}^G_H M$ to $\mathrm{Tr}^G_H f : M' \to M$ is k-linear, so it defines a k-module homomorphism

$$\mathrm{Tr}^G_H : \mathrm{Hom}_{kH}(\mathrm{Res}^G_H M', \mathrm{Res}^G_H M) \to \mathrm{Hom}_{kG}(M', M).$$

This construction extends to cochain complexes:

Proposition 62. *Let $F_* \to k$ be a free kG-resolution of k, and M is a kG-module. Then the map that sends each homomorphism $f \in \mathrm{Hom}_{kH}(\mathrm{Res}^G_H F_n, \mathrm{Res}^G_H M)$ to $\mathrm{Tr}^G_H f \in \mathrm{Hom}_{kG}(F_n, M)$ defines a chain map*

$$\mathrm{Tr}^G_H : \mathrm{Hom}_{kH}(\mathrm{Res}^G_H F_*, M) \to \mathrm{Hom}_{kG}(F_*, M).$$

The induced homomorphism

$$\mathrm{Tr}^G_H : H^n(H; \mathrm{Res}^G_H M) \to H^n(G, M)$$

is called the **transfer map** *for group cohomology.*

Proof. Given a kG-module homomorphism $\varphi : M' \to M''$, there is a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{kH}(\mathrm{Res}^G_H M'', \mathrm{Res}^G_H M) & \xrightarrow{\mathrm{Tr}^G_H} & \mathrm{Hom}_{kG}(M'', M) \\ \Big\downarrow {\scriptstyle (\mathrm{Res}^G_H \varphi)^*} & & \Big\downarrow {\scriptstyle \varphi^*} \\ \mathrm{Hom}_{kH}(\mathrm{Res}^G_H M', \mathrm{Res}^G_H M) & \xrightarrow{\mathrm{Tr}^G_H} & \mathrm{Hom}_{kG}(M', M) \end{array}$$

because for every $f : \mathrm{Res}^G_H M'' \to \mathrm{Res}^G_H M$ and for every $m \in M'$, we have

$$(\mathrm{Tr}^G_H \circ (\mathrm{Res}^G_H \varphi)^*(f))(m) = \sum_{i=1}^{k} g_i (f \circ \varphi)(g_i^{-1} m) = \sum_{i=1}^{k} g_i f(g_i^{-1} \varphi(m)) = (\varphi^* \circ \mathrm{Tr}^G_H(f))(m).$$

This shows in particular that the transfer maps commute with the boundary maps $\partial_n : F_n \to F_{n-1}$. Hence

$$\mathrm{Tr}^G_H : \mathrm{Hom}_{kH}(\mathrm{Res}^G_H(F_*), \mathrm{Res}^G_H M) \to \mathrm{Hom}_{kG}(F_*, M).$$

is a chain map. □

The composition of restriction and transfer maps is very computable when H has a finite index in G.

Proposition 63. *Let H be a subgroup of G with finite index. Then for every kG-module M, the composition*

$$H^n(G; M) \xrightarrow{\mathrm{Res}^G_H} H^n(H, \mathrm{Res}^G_H M) \xrightarrow{\mathrm{Tr}^G_H} H^n(G; M)$$

is equal to the homomorphism defined by $u \to |G : H| u$.

Proof. We prove this in the cochain level. For $f \in \mathrm{Hom}_{kG}(F_*, M)$, we have

$$(\mathrm{Tr}^G_H \mathrm{Res}^G_H f)(x) = \sum_{i=1}^{k} g_i (\mathrm{Res}^G_H f)(g_i^{-1} x) = \sum_{i=1}^{k} f(x) = k f(x).$$

So the composition $\mathrm{Tr}^G_H \circ \mathrm{Res}^G_H$ is equal to the homomorphism defined by $f \to kf$. Hence the same is true for the induced maps on cohomology. □

There are some interesting consequences of this result.

Proposition 64. *Let G be a finite group and M be a kG-module. Then $|G| \cdot H^n(G; M) = 0$ for every $n \geq 1$.*

Proof. First observe that if $G = 1$, then k is a free kG-module. So $0 \to k \xrightarrow{\varepsilon} k \to 0$ is a free resolution of k as a kG-module. From this we obtain that

$$H^n(\{1\}; M) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

This gives that the composition

$$H^n(G; M) \xrightarrow{\mathrm{Res}^G_{\{1\}}} H^n(\{1\}; M) \xrightarrow{\mathrm{Tr}^G_{\{1\}}} H^n(G; M)$$

is equal to 0 when $n \geq 1$. By Proposition 63, we conclude that $|G| \cdot H^n(G, M) = 0$ for $n \geq 1$. □

Corollary 65. *Let G be a finite group. Assume that $|G|$ is invertible in k. Then for every kG-module M, $H^n(G;M) = 0$ for $n \geq 1$.*

Example 66. Let G be a finite group. If k is a characteristic zero field such as $\mathbb{R}$ or $\mathbb{C}$, then for every kG-module M,

$$H^n(G;M) = \begin{cases} M^G & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

Another example of vanishing group cohomology is the following: Let k be a field with characteristic p. Then for every finite group G with order coprime to p, we have $H^n(G;k) = 0$ for $n \geq 1$. For example, $H^n(C_3;\mathbb{F}_2) = 0$ for $n \geq 1$.

Now we will describe a decomposition theorem for group cohomology over integral group ring. The cohomology of a finite group G with coefficients in a $\mathbb{Z}G$-module M is defined by

$$H^n(G;M) = H^n(\mathrm{Hom}_{\mathbb{Z}G}(F_*, M))$$

where $F_* \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. The free resolution $F_* \to \mathbb{Z}$ can be taken as the standard resolution. In this case for each $n \geq 0$, F_n is a finitely generated free $\mathbb{Z}G$-module. If M is also finitely generated, then we conclude that the group $\mathrm{Hom}_{\mathbb{Z}G}(F_n, \mathbb{Z})$ is a finitely generated abelian group. This gives that $H^n(\mathrm{Hom}_{\mathbb{Z}G}(F_*, \mathbb{Z}))$ is a finitely generated abelian group.

Now we are ready to state a result which gives a decomposition for group cohomology. For a finite abelian abelian group A, the abelian group $A_{(p)}$ is the direct sum of summands of A whose exponent is a power of p.

Proposition 67. *Let G be a finite group and M be a finitely generated $\mathbb{Z}G$-module. Then for $n \geq 1$, the cohomology group $H^n(G;M)$ is a finite abelian group with exponent dividing the order of G. Hence we can write*

$$H^n(G;M) \cong \bigoplus_{p \,|\, |G|} H^n(G;M)_{(p)}$$

Proof. By the above discussion $H^n(G;M)$ is a finitely generated abelian group. By Corollary 64, $|G| \cdot H^n(G;M) = 0$, for $n \geq 1$. Hence $H^n(G;M)$ is a finite abelian group with exponent dividing the order of G. Every finite abelian group with these properties has a decomposition as above. □

By the properties of tensor product, we have $A_{(p)} = A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ where $\mathbb{Z}_{(p)}$ denotes the ring of p-local integers. For each summand in the above decomposition, we have the following:

Proposition 68. *Let G be a group, M be a $\mathbb{Z}G$-module, and p be a prime number. Then for every $n \geq 0$, we have*

$$H^n(G,M)_{(p)} \cong H^n(G; M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$$

Proof. Since $\mathbb{Z}_{(p)}$ is flat over $\mathbb{Z}$, we have

$$H^n(G;M)_{(p)} \cong H^n(G;M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong H^n(\mathrm{Hom}_{\mathbb{Z}G}(F_*, M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}).$$

We also have

$$\mathrm{Hom}_{\mathbb{Z}G}(F_*, M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}G}(F_* \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}).$$

So we conclude that

$$H^n(G; M)_{(p)} \cong H^n(G; M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$$

where the cohomology group on the right is the cohomology of G with coefficients in a $\mathbb{Z}_{(p)}G$-module. □

Remark 69. Let k be a commutative ring. Given a kG-module we defined n-cohomology of G with coefficients in M by

$$H^n(G; M) = H^n(\mathrm{Hom}_{kG}(P_*, M))$$

where P_* is a projective resolution of k as a kG-module. Note that any kG-module is an abelian group, so it can be considered in natural way as a $\mathbb{Z}G$-module. The cohomology of G with coefficient in M considered as a $\mathbb{Z}G$-module is isomorphic to the cohomology when M is considered as a kG-module. This is because if $F_* \to \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, then $F_* \otimes_{\mathbb{Z}} k \to k$ is a free resolution of k as a kG-module, and there is an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}G}(F_*, {}_{\mathbb{Z}}M) \cong \mathrm{Hom}_{kG}(F_* \otimes_{\mathbb{Z}} k, M)$$

of cochain complexes. So their cohomology groups are isomorphic.

Restriction and Transfer for Homology

Let G be a group and $H \leq G$ be a subgroup of G with finite index. Suppose that k is a commutative ring and M is a (left) kG-module. Given a free resolution $F_* \to k$ of k as a kG-module, $\mathrm{Res}^G_H F_*$ is a free resolution of k as a kH-module. By Lemma 25, we have

$$F_* \otimes_{kG} M \cong (F_* \otimes_k M)_G \quad \text{and} \quad \mathrm{Res}^G_H F_* \otimes_{kH} \mathrm{Res}^G_H M \cong (\mathrm{Res}^G_H F_* \otimes_k \mathrm{Res}^G_H M)_H.$$

Recall that to define the tensor product $F_* \otimes_{kG} M$, we consider F_* as a right kG-module via the right action given by $xg = g^{-1}x$ for $x \in F_n$. The G-action on $F_* \otimes_k M$ is taken to be the diagonal action $g(x \otimes m) = gx \otimes gm$.

Since $I_H \subseteq I_G$, for every kG-module N, there is a k-module homomorphism $N_H = N/I_H N \to N_G = N/I_G N$. This gives an abelian group homomorphism

$$\mathrm{Res}^G_H : \mathrm{Res}^G_H F_n \otimes_{kH} \mathrm{Res}^G_H M \to F_n \otimes_{kG} M$$

for each $n \geq 0$. By the naturality of this homomorphism, we obtain the following:

Proposition 70. *Let $H \leq G$ be a subgroup of finite index and M be a kG-module. Then there is a chain map*

$$\mathrm{Res}^G_H : \mathrm{Res}^G_H F_* \otimes_{kH} \mathrm{Res}^G_H M \to F_* \otimes_{kG} M.$$

The induced map on homology

$$\mathrm{Res}^G_H : H_n(H; \mathrm{Res}^G_H M) \to H_n(G; M)$$

is called the **restriction map** *for group homology.*

To construct a map on the other direction we need to show that for every kG-module N, there is a homomorphism $\mathrm{Tr}_H^G : N_G \to N_H$. Let $H\backslash G = \{Hg_1, \dots, Hg_k\}$ be a set of coset representatives. For every $n \in N$, let $\mathrm{Tr}_H^G : N_G \to N_H$ be the map defined by

$$\mathrm{Tr}_H^G([n]_G) = \Big[\sum_{i=1}^{k} g_i n\Big]_H.$$

We need to show that this map is well-defined. Let $H\backslash G = \{Hg'_1, \dots, Hg'_k\}$ be another set of coset representatives. Then for every i, there is an element $h_i \in H$ such that $g'_i = h_i g_i$. This gives

$$\Big[\sum_{i=1}^{k} g'_i n\Big]_H = \Big[\sum_{i=1}^{k} h_i g_i n\Big]_H = \Big[\sum_{i=1}^{k} g_i n + \sum_{i=1}^{k}(h_i - 1) g_i n\Big]_H = \Big[\sum_{i=1}^{k} g_i n\Big]_H.$$

So the definition of $\mathrm{Tr}_H^G([n]_G)$ does not depend on the chosen set of coset representatives. Now we will show that $\mathrm{Tr}_H^G([n]_G)$ does not depend on the representative $n \in N$. Note that if $n = (1-g)n'$ for some $g \in G$ and $n' \in N$, then

$$\sum_{i=1}^{k} g_i n = \sum_{i=1}^{k} g_i (1-g) n' = \sum_{i=1}^{k} g_i n' - \sum_{i=1}^{k} g_i g n'$$

Since $\{Hg_1 g, \dots, Hg g_n\}$ is another set of coset representatives, we obtain that $\sum_{i=1}^{k} g_i n$ lies in I_H. This show that $\mathrm{Tr}_H^G : N_G \to N_H$ is well-defined.

Applying the transfer map constructed above to $F_* \otimes_{kG} M \cong (F_* \otimes_k M)_G$, we obtain:

Proposition 71. *Let $H \leq G$ be a subgroup of finite index and M be a kG-module. Then there is a chain map*

$$\mathrm{Tr}_H^G : F_* \otimes_{kG} M \to \mathrm{Res}_H^G F_* \otimes_{kH} \mathrm{Res}_H^G M.$$

The induced map on homology

$$\mathrm{Tr}_H^G : H_n(G; M) \to H_n(H; \mathrm{Res}_H^G M)$$

is called the **transfer map** *for group homology.*

Exercise 72. Show that the composition

$$\mathrm{Res}_H^G \mathrm{Tr}_H^G : H_n(G; M) \to H_n(H; \mathrm{Res}_H^G) \to H_n(G; M)$$

is equal to the map $u \to |G : H|u$. Conclude that if G is a finite group, then for every $n \geq 1$, $|G| \cdot H_n(G, M) = 0$.

4.2 Functoriality of Group (Co)homology

Functoriality of Group Cohomology

Let k be a commutative ring and $f : H \to G$ be a group homomorphism. Every kG-module M can be viewed as a kH-module via the H-action defined by $h \cdot m = f(h)m$ for all $h \in H$ and $m \in M$. We denote this kH-module by $\mathrm{Res}_f M$.

In this section we use the notation $H^*(G; M)$ for cohomology of G with coefficients in a kG-module M. $H^*(G; M)$ denotes the direct sum $\bigoplus_{n=0}^{\infty} H^n(G; M)$ as a graded k-module.

Proposition 73. *Let $f : H \to G$ be a group homomorphism. Then for every kG-module M, the homomorphism f induces a graded k-module homomorphism*

$$f^* : H^*(G; M) \to H^*(H; \mathrm{Res}_f M).$$

Proof. Let $P_* \to k$ be an kG-projective resolution of k, and $Q_* \to k$ be an kH-projective resolution of k. Since

$$\cdots \xrightarrow{\partial_2} \mathrm{Res}_f P_1 \xrightarrow{\partial_1} \mathrm{Res}_f P_0 \to k \to 0$$

is an exact sequence of kH-modules (not necessarily a kH-projective resolution), there is a chain map $\mu_* : Q_* \to \mathrm{Res}_f P_*$ covering the identity map on k. This gives a chain map

$$\mu^* : \mathrm{Hom}_{kG}(P_*, M) \to \mathrm{Hom}_{kH}(Q_*, \mathrm{Res}_f M)$$

defined by $\mu^*(\varphi) = \mathrm{Res}_f \varphi \circ \mu_*$ for every $\varphi : P_* \to M$. The induced map on cohomology gives the desired homomorphism $f^* : H^*(G; M) \to H^*(H; \mathrm{Res}_f)$. □

In the special case where $f : H \to G$ is an inclusion of a subgroup, then the induced map f^* coincides with the restriction map $\mathrm{Res}^G_H : H^*(G; M) \to H^*(H; \mathrm{Res}^G_H M)$.

Proposition 74. *If $\xi : M \to M'$ is a kG-module homomorphism, then for every kG-projective resolution $P_* \to k$, there is a induced homomorphism*

$$\mathrm{Hom}_{kG}(P_*, M) \to \mathrm{Hom}_{kG}(P_*, M')$$

defined by composition with ξ. This induces a homomorphism $\xi_* : H^*(G; M) \to H^*(G; M')$ of graded k-modules.

It is possible to combine the above induced maps to establish the group cohomology $H^*(G; M)$ as a functor from a category of pairs to the category of graded k-modules. Let $\mathcal{D}$ be the category whose objects are the pairs (G, M) where G is a group and M is a kG-module. A morphism $(G, M) \to (G', M')$ in $\mathcal{D}$ is given by a pair (f, ξ) where $f : G \to G'$ is a group homomorphism and $\xi : \mathrm{Res}_f M' \to M$ is a kG-module homomorphism. The group cohomology $H^*(G; M)$ defines a contravariant functor from the category $\mathcal{D}$ to the category of graded k-modules. For each morphism $(f, \xi) : (G, M) \to (G', M')$, the homomorphism $H^*(f, \xi) : H^*(G'; M') \to H^*(G; M)$ is defined to be the composition

$$H^*(G', M') \xrightarrow{f^*} H^*(G; \mathrm{Res}_f M') \xrightarrow{\xi_*} H^*(G; M).$$

Conjugation Map for Group Cohomology

An important example of a homomorphism in group cohomology is the conjugation map. Let $H \leq G$ be a subgroup of G and $g \in G$ be an element. The sets

$${}^gH = \{ghg^{-1} \in G \,|\, h \in H\} \text{ and } H^g = \{g^{-1}hg \,|\, h \in H\}$$

are subgroups of G with usual multiplication in G. Consider the group homomorphism

$$c_{g,H} : H \to {}^gH$$

defined by $c_{g,H}(h) = ghg^{-1}$. We call the homomorphism $c_{g,H}$ the homomorphism induced by conjugation. For every $k[{}^gH]$-module M, the conjugation map induces a graded k-module homomorphism

$$c_{g,H}^* : H^*({}^gH; M) \to H^*(H; \mathrm{Res}_{c_{g,H}} M).$$

Let M be a kG-module. Then the group homomorphism $c_{g^{-1},{}^gH} : {}^gH \to H$ gives a homomorphism

$$c_{g^{-1},{}^gH}^* : H^*(H; \mathrm{Res}_H^G M) \to H^*({}^gH; M')$$

where $M' = \mathrm{Res}_{c^{-1},{}^gH} \mathrm{Res}_H^G M$ is the $k[{}^gH]$-module with the ${}^gH$-action defined by $(ghg^{-1})m = hm$ for all $h \in H$ and $m \in M$. Consider the k-module homomorphism $\xi_g : M' \to \mathrm{Res}_{{}^gH}^G M$ defined by $\xi_g(m) = gm$ for all $m \in M$. For every $h \in H$, we have

$$\xi_g((ghg^{-1})m) = \xi_g(hm) = ghm = (ghg^{-1})\xi(m).$$

So, ξ_g is a $k[{}^gH]$-module homomorphism. This gives a homomorphism of graded k-modules

$$H^*({}^gH; M') \xrightarrow{(\xi_g)_*} H^*({}^gH; \mathrm{Res}_{{}^gH}^G M).$$

Definition 75. Let $H \leq G$ be a subgroup of G and $g \in G$. For every kG-module M, the composition

$$C_{g,H} : H^*(H; \mathrm{Res}_H^G M) \xrightarrow{c_{g^{-1},{}^gH}^*} H^*({}^gH; M') \xrightarrow{(\xi_g)_*} H^*({}^gH; \mathrm{Res}_{{}^gH}^G M).$$

is called the **conjugation map** in group cohomology.

In Brown [2], for every $u \in H^*(H; \mathrm{res}_H^G M)$, the element $C_{g,H}(u)$ is denoted by gu. Note that the pair $(c_{g^{-1},{}^gH}, \xi_g)$ defines a morphism

$$(c_{g^{-1},{}^gH}, \xi_g) : ({}^gH, \mathrm{Res}_{{}^gH}^G M) \to (H, \mathrm{Res}_H^G M)$$

in the category $\mathcal{D}$, and the homomorphism $C_{g,H}$ is the homomorphism induced by this morphism in $\mathcal{D}$. On the chain level the homomorphism induced by the pair $(c_{g^{-1},{}^gH}, \xi_g)$ has much more simpler formula. We now explain this formula.

Let $P_* \to k$ be a kG-projective resolution of k and M be a kG-module. For every $H \leq G$ and $g \in G$, let $Q_* \to k$ denote the $k[{}^gH]$-projective resolution obtained as $\mathrm{Res}_{c_{g^{-1},{}^gH}} \mathrm{Res}_H^G P_*$. There is chain map of $k[{}^gH]$-modules $\mu_* : \mathrm{Res}_{{}^gH}^G P_* \to Q_*$ covering the identity map on k. We can take this chain map (up to chain homotopy) as the chain map defined by $\mu_*(x) = g^{-1}x$ for every $x \in P_*$. Note that $\mu_*((ghg^{-1})x) = hg^{-1}x = (ghg^{-1}) \cdot \mu_*(x)$ for every $h \in H$ and $x \in P_*$, so μ_* is a chain map of $k[{}^gH]$-modules.

Lemma 76. *Consider the chain map induced by the pair* $(c_{g^{-1},{}^gH}, \xi_g)$

$$\mathrm{Hom}_{kH}(\mathrm{Res}_H^G P_*, M) \xrightarrow{c_{g^{-1},{}^gH}} \mathrm{Hom}_{k[{}^gH]}(\mathrm{Res}_{{}^gH}^G P_*, M') \to \mathrm{Hom}_{k[{}^gH]}(\mathrm{Res}_{{}^gH}^G P_*, M)$$

where the first map is defined by $\mu^*(\varphi) = \mathrm{Res}_{c_{h^{-1},{}^gH}}(\varphi) \circ \mu_*$ *for every* $\varphi : \mathrm{Res}_H^G P_* \to M$, *and the second map is given by composition with* $\xi_g : M' \to M$. *Then for every* kH*-module homomorphism* $\varphi : \mathrm{Res}_H^G P_* \to M$ *and for* $x \in P_*$, *we have*

$$(c_{g^{-1},{}^gH}, \xi_g)^*(\varphi)(x) = g\varphi(g^{-1}y).$$

Proof. This is clear from the definitions given above. $\square$

We will use Lemma 76 later in the proof of Mackey formula for group cohomology. Another consequence is the following:

Proposition 77. *Let $H \leq G$ be a subgroup and M be a $\mathbb{Z}G$-module. Then for every $h \in H$, the homomorphism*

$$C_{h,H} : H^*(H; \mathrm{res}^G_H M) \to H^*(H; \mathrm{Res}^G_H M)$$

is equal to the identity map.

Proof. We will show this on the chain level. By Lemma 76, for every kH-module homomorphism $\varphi : \mathrm{Res}^G_H P_* \to M$ and for $x \in P_*$, we have

$$(c_{h^{-1},H}, \xi_h)^*(\varphi)(x) = h\varphi(h^{-1}x).$$

Since φ is a kH-module homomorphism, the last expression is equal to x. This gives $(c_{h^{-1},H}, \xi_h)^*(\varphi) = \mathrm{id}$, hence $C_{h,H} = \mathrm{id}$. $\square$

The following observation is useful for calculations involving the conjugation map.

Lemma 78. *Let $L \leq H \leq G$ be subgroups of G and $g \in G$. Then for every kG-module M, the following diagram commutes:*

$$\begin{array}{ccc} H^*(H; \mathrm{Res}^G_H M) & \xrightarrow{C_{g,H}} & H^*({}^gH; \mathrm{Res}^G_{{}^gH} M) \\ \downarrow{\scriptstyle \mathrm{Res}^H_L} & & \downarrow{\scriptstyle \mathrm{Res}^{{}^gH}_{{}^gL}} \\ H^*(L; \mathrm{Res}^G_L M) & \xrightarrow{C_{g,L}} & H^*({}^gL; \mathrm{Res}^G_{{}^gL} M). \end{array}$$

Proof. This follows from Lemma 76. $\square$

Functoriality of Group Homology

Functoriality of group homology is defined in a similar way. Let $f : H \to G$ be a group homomorphism. For every kG-module M, let $\mathrm{Res}_f M$ denote the kH-module defined by $h \cdot m = f(h)m$ for all $h \in H$ and $m \in M$. $H_*(G; M)$ denotes the direct sum $\bigoplus_{n=0}^{\infty} H_n(G; M)$ as a graded k-module.

Proposition 79. *Let $f : H \to G$ be a group homomorphism. Then for every kG-module M, the homomorphism f induces a graded k-module homomorphism*

$$f_* : H_*(H; \mathrm{Res}_f M) \to H_*(G; M).$$

Proof. Let $P_* \to k$ be a projective resolution of k as a right kG-module, and $Q_* \to k$ be a projective resolution of k as a right kH-module. Let $\mu_* : Q_* \to \mathrm{Res}_f P_*$ be the chain map covering the identity map on k. This gives a chain map

$$Q_* \otimes_{kH} \mathrm{Res}_f M \to \mathrm{Res}_f P_* \otimes_{kH} \mathrm{Res}_f M \to P_* \otimes_{kG} M$$

defined by

$$[x \otimes m]_H \to [\mu_*(x) \otimes m]_H \to [\mu_*(x) \otimes m]_G$$

for every $x \in P_*$ and $m \in M$. The induced map on cohomology gives the desired homomorphism $f_* : H_*(H; \mathrm{Res}_f M) \to H_*(G; M)$. □

If $\xi : M \to M'$ is a kG-module homomorphism, then for every kG-projective resolution $P_* \to k$, there is a induced homomorphism

$$P_* \otimes_{kG} M \to P_* \otimes_{kG} M'$$

defined by composition with ξ. This induces a homomorphism $\xi_* : H^*(G; M) \to H^*(G; M')$ of graded k-modules.

As in the case of cohomology it is possible to combine the above induced maps. Let $\mathcal{C}$ be the category of pairs whose objects are the pairs (G, M) where G is a group and M is a kG-module. A morphism $(G, M) \to (G', M')$ in $\mathcal{C}$ is given by a pair (f, ξ) where $f : G \to G'$ is a group homomorphism and $\xi : M \to \mathrm{Res}_f M'$ is a kG-module homomorphism. The group homology $H_*(G; M)$ defines a covariant functor from the category $\mathcal{C}$ to the category of graded k-modules. For every morphism $(f, \xi) : (G, M) \to (G', M')$, the homomorphism $H_*(f, \xi) : H_*(G; M) \to H_*(G'; M')$ is defined to be the composition

$$H_*(G, M) \xrightarrow{\xi_*} H^*(G; \mathrm{Res}_f M') \xrightarrow{f_*} H^*(G' : M').$$

Exercise 80. Define conjugation maps for group homology. State and prove an analogue of Lemma 76 and Proposition 77 for group homology.

4.3 Shapiro's Lemma

Shapiro's Lemma for Cohomology

Let G be a group and $H \leq G$ be a subgroup of G. The group ring kG can be considered as a kH-kG-bimodule with left and right multiplication.

Definition 81. For every kH-module M, the kG-module

$$\mathrm{Coind}_H^G M := \mathrm{Hom}_{kH}(kG, M)$$

is called the **coinduced module**. The additive functor $\mathrm{Coind}_H^G(-) : RH{-}Mod \to RG{-}Mod$ is called the **coinduction functor**.

By adjointness of tensor and Hom-functors, for every kG-module N and kH-module M, we have

$$\mathrm{Hom}_{kG}(N, \mathrm{Coind}_H GM) \cong \mathrm{Hom}_{kG}(N, \mathrm{Hom}_{kH}(kG, M)) \cong \mathrm{Hom}_{kH}(N \otimes_{kG} kG, M)$$

where in the last expression, $N \otimes_{kG} kG$ is considered as a left kH-module via the H-action defined by $h(n \otimes x) = n \otimes xh^{-1}$. Since $N \otimes_{kG} kG \cong N$ as k-modules, this kH-module is $\mathrm{Res}_H^G N$. We obtain the following:

Proposition 82. *The Coinduction functor* $\mathrm{Coind}_H^G(-)$ *is right adjoint to the induction functor* $\mathrm{Res}_H^G(-)$*, i.e., there is a natural isomorphism*

$$\mathrm{Hom}_{kH}(\mathrm{Res}_H^G N, M) \cong \mathrm{Hom}_{kG}(N, \mathrm{Coind}_H^G M).$$

Now we are ready to prove Shapiro's lemma.

Theorem 83 (Shapiro's Lemma for Group Cohomology)**.** *Let* G *be a group and* H *be a subgroup of* G*. For every* kH*-module* M*, there is an isomorphism*

$$\Phi : H^n(G; \mathrm{Coind}_H^G M) \xrightarrow{\cong} H^n(H; M).$$

Proof. Let $F_* \to k$ be a free resolution of k as a kG-module. Then by Proposition 82, there is an isomorphism of chain complexes

$$\phi_* : \mathrm{Hom}_{kG}(F_*, \mathrm{Coind}_H^G M) \xrightarrow{\cong} \mathrm{Hom}_{kH}(\mathrm{Res}_H^G F_*; M).$$

Since $\mathrm{Res}_H^G F_*$ is a free resolution of k as a kH-module, the cohomology groups of the right hand side is isomorphic to $H^n(H; M)$. So the isomorphism above gives the desired isomorphism on cohomology groups. □

The isomorphism in Shapiro's lemma is connected to the restriction map on cohomology.

Proposition 84. *Let* M *be a* kG*-module and* $\mu : M \to \mathrm{Coind}_H^G \mathrm{Res}_H^G M$ *be the* kG*-module homomorphism defined by* $\mu(m)(g) = gm$ *for* $m \in M$ *and* $g \in G$*. Then there is a commuting diagram*

$$\begin{array}{ccc} H^n(G; \mathrm{Coind}_H^G \mathrm{Res}_H^G M) & \xrightarrow{\Phi \cong} & H^n(H; \mathrm{Res}_H^G M) \\ & {\scriptstyle \mu_*} \nwarrow \quad \nearrow {\scriptstyle \mathrm{Res}_H^G} & \\ & H^n(G; M) & \end{array}$$

where μ_* *is the homomorphism induced by* μ*.*

Proof. We will show that there is a similar commuting diagram on the cochain level. Let $F_* \to k$ be a free kG-resolution of k. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{kG}(F_*, \mathrm{Coind}_H^G \mathrm{Res}_H^G M) & \xrightarrow{\phi_*} & \mathrm{Hom}_{kH}(\mathrm{Res}_H^G F_*, \mathrm{Res}_H^G M) \\ & {\scriptstyle \mu_*} \nwarrow \quad \nearrow {\scriptstyle \mathrm{Res}_H^G} & \\ & \mathrm{Hom}_{kG}(F_*, M) & \end{array}$$

where μ_* is defined by composition with μ. Let $f : F_* \to M$ be a kG-module homomorphism. Then for $x \in F_n$, we have

$$(\phi_* \circ \mu_*)(f)(x) = [\phi_*(\mu \circ f)](x) = (\mu(f(x))(1) = f(x).$$

Hence the diagram commutes and the proof is complete. □

As a corollary of Shapiro's lemma we obtain the following:

Proposition 85. *Let P be a projective kG-module. Then $H^n(G; P) = 0$ for $n \geq 0$.*

Proof. It is enough to prove this for $P = kG$ since group cohomology respects direct sums, i.e., if $M \cong \bigoplus_i M_i$, then $H^n(G; M) \cong \bigoplus_i H^n(G; M_i)$. Note that $kG \cong \mathrm{Ind}_{\{1\}}^G k$. Then by Shapiro's lemma we have

$$H^n(G; kG) \cong H^n(\{1\}; k) = 0$$

for $n \geq 1$. □

Shapiro's Lemma for Homology

There is a Shapiro's isomorphims for group homology similar to the one for cohomology. For this isomoprhism we used induced modules instead of coinduced modules.

Let G be a group and $H \leq G$ be a subgroup of G. The group ring kG can be considered as a kG-kH-bimodule with left and right multiplication.

Definition 86. For every kH-module M, the kG-module

$$\mathrm{Ind}_H^G M := kG \otimes_{kH} M$$

is called the **induced module**. The additive functor $\mathrm{Ind}_H^G(-) : RH - Mod \to RG - Mod$ is called the **induction functor**.

Note that $\mathrm{Ind}_H^G M$ is considered as a left kG-module with the G-action given by $g(g' \otimes m) = gg' \otimes m$. In the Shapiro's lemma for cohomology we use the adjointness property of tensor and Hom-functors. For homology we will be using the following lemma:

Lemma 87. *Let N be a kG-module and M be a kH-module. Then there is an isomorphism*

$$N \otimes_{kG} \mathrm{Ind}_H^G M \cong \mathrm{Res}_H^G N \otimes_{kH} M.$$

Proof. Consider the homomorphism $\theta : N \otimes_{kG} (kG \otimes_{kH} M) \to \mathrm{Res}_H^G N \otimes_{kH} M$ defined by $[n \otimes g \otimes m] \to [ng \otimes M]$. The inverse of θ is given by $[n \otimes m] \to [n \otimes 1 \otimes m]$, so θ is an isomorphism. □

Now we are ready to state Shapiro's lemma for homology.

Theorem 88 (Shapiro's Lemma for Homology)**.** *Let G be a group and H be a subgroup of G. For every kH-module M, there is an isomorphism*

$$\Phi : H_n(G; \mathrm{Ind}_H^G M) \xrightarrow{\cong} H_n(H; M).$$

Proof. Let $F_* \to k$ be a free resolution of k as a kG-module. Then by Lemma 87, there is an isomorphism of chain complexes

$$\phi_* : F_* \otimes_{kG} \mathrm{Ind}_H^G M \xrightarrow{\cong} \mathrm{Res}_H^G F_* \otimes_{kH} M.$$

Since $\mathrm{Res}_H^G F_*$ is a free resolution of k as a kH-module, the homology groups of the right hand side is isomorphic to $H_n(H; M)$. So the chain map ϕ_* induces the desired isomorphism on homology groups. □

As in the cohomology case, Shapiro's isomorphism is related to the restriction map on homology.

Proposition 89. *Let M be a kG-module and $\eta : \mathrm{Ind}_H^G \mathrm{Res}_H^G M \to M$ be the kG-module homomorphism defined by $\eta(g \otimes m) = gm$ for $m \in M$ and $g \in G$. Then there is a commuting diagram*

$$\begin{array}{ccc} H_n(G; \mathrm{Ind}_H^G \mathrm{Res}_H^G M) & \xrightarrow{\Phi \cong} & H_n(H; \mathrm{Res}_H^G M) \\ & {\scriptstyle \eta_*} \searrow \quad \swarrow {\scriptstyle \mathrm{Res}_H^G} & \\ & H_n(G; M) & \end{array}$$

where η_* is the homomorphism induced by η.

Proof. As in the cohomology case, we will show that there is a similar commuting diagram on the chain complex level. Let $F_* \to k$ be a free kG-resolution of k. Consider the diagram

$$\begin{array}{ccc} F_* \otimes_{kG} \mathrm{Ind}_H^G \mathrm{Res}_H^G M & \xrightarrow{\phi_*} & \mathrm{Res}_H^G F_* \otimes_{kH} \mathrm{Res}_H^G M \\ & {\scriptstyle \eta_*} \searrow \quad \swarrow {\scriptstyle \mathrm{Res}_H^G} & \\ & F_* \otimes_{kG} M & \end{array}$$

where η_* is defined by composition with η. For $x \in F_n$ and $m \in M$, we have

$$(\mathrm{Res}_H^G \circ \phi_*)(x \otimes (g \otimes m)) = xg \otimes m = x \otimes gm = x \otimes \eta(g \otimes m).$$

Hence the diagram commutes and the proof is complete. □

We have the following corollary of Shapiro's lemma.

Proposition 90. *Let P be a projective kG-module. Then $H_n(G; P) = 0$ for $n \geq 0$.*

Proof. Proof is similar to the cohomology case. □

4.4 Cartan-Eilenberg Theorem

Let G be a finite group and M be a $\mathbb{Z}G$-module. We have seen in Section 4.1 that for every $n \geq 0$,

$$H^n(G, M) \cong \bigoplus_{p \,|\, |G|} H^n(G, M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}).$$

This reduces the computation of the cohomology of G to the computation of $H^n(G; M \oplus_{\mathbb{Z}} \mathbb{Z}_{(p)})$ for all the primes p dividing the order of G. Because of this in this section we assume k is a commutative ring such that all the integers coprime to p are invertible in k. We call such a ring a p-local ring. Examples of a p-local ring are p-local integers $\mathbb{Z}_{(p)}$ or a field $\mathbb{F}$ with p-elements.

Lemma 91. *Let H be a subgroup of G such that $|G : H|$ is coprime to p. Assume that k is a p-local ring. Then for every kG-module M, the restriction map*

$$\mathrm{Res}_H^G : H^n(G; M) \to H^n(H; \mathrm{Res}_H^G M)$$

is a monomorphism.

Proof. Let $u \in H^n(G; M)$ such that $\text{Res}^G_H u = 0$. By Proposition 63, we have

$$|G : H|u = \text{Tr}^G_H \text{Res}^G_H(u) = 0.$$

Since $|G : H|$ is coprime to p, it is invertible in k. Hence $u = |G : H|^{-1}|G : H|u = 0$. □

If S is a Sylow p-subgroup of G, then $\text{Res}^G_S : H^n(G; M) \to H^n(S; \text{Res}^G_S M)$ is a monomorphism. Cartan-Eilenberg theorem identifies the image of the restriction map to the subgroup of stable elements. Before we introduce the Cartan-Eilenberg theorem, we need to prove another important formula for restriction and transfer maps.

Proposition 92.

1. *For every $K \le H \le G$ such that $|G : K|$ is finite, we have*

$$\text{Res}^H_K \circ \text{Res}^G_H = \text{Res}^G_K \quad \textit{and} \quad \text{Tr}^G_H \circ \text{Tr}^H_K = \text{Tr}^G_K.$$

2. *For every $H, K \le G$ with $|G : H|$ finite, we have*

$$\text{Res}^G_K \circ \text{Tr}^G_H = \sum_{HgK \in H\backslash G/K} \text{Tr}^K_{K\cap H^g} \circ \text{Res}^{{}^gH}_{K\cap {}^gH} \circ C_{g,H}$$

where $H\backslash G/K$ is the set of double cosets $\{HgK \mid g \in G\}$ of H and K in G.

Proof. (1) The equation for restriction follows easily from the definitions. For the equation for transfer, observe that if $G/H = \{g_1H, \ldots, g_kH\}$ and $H/K = \{h_1K, \ldots, h_mK\}$ are sets of cosets representatives, then a set of coset representatives for G/K can be take as $\{g_ih_jK \mid i = 1, \ldots, k, j = 1, \ldots, m\}$. Then for every homomorphism $f_* \in \text{Hom}_{kK}(\text{Res}^G_K F_*, \text{Res}^G_K M)$, and $x \in F_*$, we have

$$(\text{Tr}^G_K f)(x) = \sum_{i,j} g_ih_jf(h_j^{-1}g_i^{-1}x) = \sum_i g_i(\text{Tr}^H_K f)(g_i^{-1}x) = (\text{Tr}^G_H \text{Tr}^H_K f)(x).$$

(2) The subgroup K acts on $G/H = \{g_1H, \ldots, g_kH\}$, and as a K-set we have

$$\text{Res}^G_K(G/H) \cong \coprod_{KgH \in K\backslash G/H} K/(K \cap {}^gH).$$

We can interpret this formula as follows: Without loss of generality, we can assume that $\{Kg_1H, \ldots, Kg_mH\}$ are the set of double coset representatives for some $m \le k$. Then

$$G \cong \coprod_{i=1}^m Kg_iH$$

as a K-H-biset. For each $i \in \{1, \ldots, m\}$, the stabilizer of g_iH as a K-set is

$$Stab_K(g_iH) = \{k \in K \mid kg_iH = g_iH\} = \{k \in K \mid g_i^{-1}kg_i \in H\} = K \cap {}^{g_i}H.$$

So, for each $i = 1, \ldots, m$, $Kg_iH \cong K/(K \cap {}^{g_i}H)$ as K-sets. For each $i = 1, \ldots, m$, let

$$J_i = \{k_1(K \cap {}^{g_i}H), \ldots, k_{l_i}(K \cap {}^{g_i}H)\}$$

be the set of coset representatives of $K \cap {}^{g_i}H$ in K.

Let $P_* \to k$ be a kG-projective resolution of k. For every kH-homomorphism φ : $\mathrm{Res}^G_H P_* \to \mathrm{Res}^G_H M$, and for every $x \in P_*$, we have

$$\begin{aligned}\mathrm{Res}^G_K \mathrm{Tr}^G_H(\varphi)(x) &= \sum_{i=1}^{k} g_i\varphi(g_i^{-1}x) = \sum_{i=1}^{m}\sum_{k_j\in J_i} k_j g_i \varphi(g_i^{-1}k_j^{-1}x)\\ &= \sum_{i=1}^{m} \mathrm{Tr}^K_{K\cap {}^{g_i}H}\mathrm{Res}^{{}^{g_i}H}_{K\cap {}^{g_i}H}(g_i\varphi g_i^{-1})(x).\end{aligned}$$

By Lemma 76, for each $i = 1, \dots, m$, we have

$$g_i\varphi g_i^{-1} = (c_{g_i^{-1}, {}^{g_i}H}, \xi_{g_i})^*(\varphi).$$

So the induced map on cohomology satisfies the desired equality.

□

Definition 93. Let $H \leq G$ be a subgroup of G, and M be a kG-module. An element $u \in H^n(H; M)$ is called G**-stable** if for every $g \in G$,

$$\mathrm{Res}^{{}^gH}_{H\cap {}^gH} \circ C_{g,H}(u) = \mathrm{Res}^{H}_{H\cap {}^gH}(u).$$

The set of stable elements forms a group. We denote this group by $H^S(G; \mathrm{Res}^G_S M)^{G\text{-Stab}}$.

Now we are ready to state and prove the Cartan-Eilenberg Theroem.

Theorem 94 (Cartan-Eilenberg Theorem)**.** *Let G be a finite group and P be a Sylow p-subgroup of G. Assume that k is a p-local ring. Then for every kG-module M, the restriction map induces an isomorphism*

$$\mathrm{Res}^G_P : H^*(G; M) \xrightarrow{\cong} H^n(P; \mathrm{Res}^G_P M)^{G\text{-}Stab}.$$

Proof. By Lemma 91, the restriction map $\mathrm{Res}^G_P : H^*(G; M) \to H^*(P; \mathrm{Res}^G_P M)$ is a monomorphism. For every $u \in H^*(G; M)$, we claim that $\mathrm{Res}^G_P(u)$ is G-stable. We have

$$\begin{aligned}(\mathrm{Res}^{{}^gP}_{P\cap {}^gP} \circ C_{g,P})(\mathrm{Res}^G_P(u)) &= \mathrm{Res}^{{}^gP}_{P\cap {}^gP} \circ (C_{g,P} \circ \mathrm{Res}^G_P)(u)\\ &= \mathrm{Res}^{{}^gP}_{P\cap {}^gP} \circ (\mathrm{Res}^G_{{}^gP} \circ C_{g,G})(u)\\ &= \mathrm{Res}^{G}_{P\cap {}^gP} \circ C_{g,G}(u)\end{aligned}$$

where the second equality follows Lemma 78 and last equality follows from Proposition 92. By Proposition 77, $C_{g,G} = \mathrm{id}_{H^*(G;M)}$, so the right hand side becomes $\mathrm{Res}^G_{P\cap {}^gP}$ which is equal to $\mathrm{Res}^P_{P\cap {}^gP} \circ \mathrm{Res}^G_P(u)$. From this we conclude that $\mathrm{Res}^G_P(u)$ is G-stable.

Now assume that $u \in H^*(P; \mathrm{Res}^G_P M)$ is G-stable. Let $w = \mathrm{Tr}^G_P(u)$. By Proposition 92, we have

$$\mathrm{Res}^G_P w = \mathrm{Res}^G_P \circ \mathrm{Tr}^G_P(u) = \sum_{PgP\in P\backslash G/P} \mathrm{Tr}^P_{P\cap {}^gP} \circ \mathrm{Res}^{{}^gP}_{P\cap {}^gP} \circ C_{g,P}(u).$$

Since u is G-stable, the RHS becomes

$$\sum_{PgP \in P\backslash G/P} \mathrm{Tr}^P_{P\cap {}^gP} \circ \mathrm{Res}^P_{P\cap {}^gP}(u) = \sum_{PgP\in P\backslash G/P} |P : P \cap {}^gP|u$$

Note that $\mathrm{Res}^G_P(G/P) = \coprod_{P\backslash G/P}(P/P \cap {}^gP)$, so we have

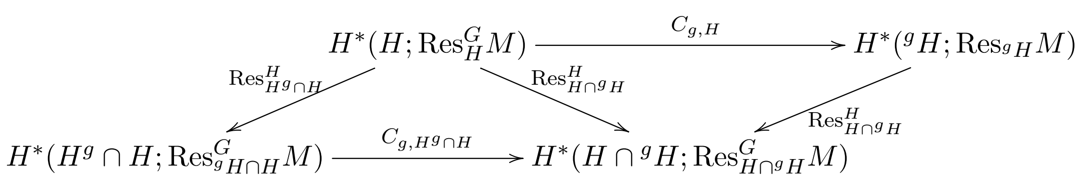
$$\coprod_{P\backslash G/P} |P/P \cap {}^gP| = |G : P|.$$

This gives

$$\mathrm{Res}^G_P u = \sum_{PgP\in P\backslash G/P} |P : P \cap {}^gP|u = |G : P|u.$$

Hence $\mathrm{Res}^G_P(|G : P|^{-1}w) = u$. We conclude that the image of the restriction map Res^G_P is equal to $H^*(P : \mathrm{Res}^G_P M)^{G\text{-Stab}}$. □

Remark 95. For $H \leq G$ and $g \in G$, consider the diagram



By Lemma 78, the outer square in the above diagram commutes, i.e., the equality

$$C_{g,H^g\cap H} \circ \mathrm{Res}^H_{H^g\cap H} = \mathrm{Res}^{{}^gH}_{H\cap {}^gH} \circ C_{g,H}$$

holds. This allows us to define the stability condition for $u \in H^*(H; \mathrm{Res}^G_H M)$ by the equation

$$C_{g,H^g\cap H} \circ \mathrm{Res}^H_{H^g\cap H} = \mathrm{Res}^H_{H\cap {}^gH}(u).$$

This version is especially used to define the G-stability in terms of the fusion system $\mathcal{F}_P(G)$ defined on a Sylow p-subgroup P induced by G.

5 Product Structure on Group Cohomology

5.1 Diagonal Approximation and Cup Product

Let G and G' be two groups and k be a commutative ring. If M is a kG-module and M' is a kG'-module then the tensor product $M \otimes_k M'$ is a $k[G \times G']$-module with the $G \times G'$-action given by $(g, g')(m \otimes m') = gm \otimes g'm'$. If $M = kX$ is a free kG-module with basis X and $M' = kX'$ is a free kG'-module with basis X', then $M \otimes_k M' = kX \otimes_k kX'$ is a free $k[G\times G']$-module with basis $X \times X'$. From this it is also easy to see that if P_* is a projective kG-module and kG' is a projective kG'-module, then $P_* \otimes_k P'_*$ is a projective $k[G \times G']$-module.

If $P_* \to k$ is a kG-projective resolution of k and $P'_* \to k$ is a kG'-projective resolution of k, then $P_* \otimes_k P'_* \to k \otimes_k k = k$ is a projective $k[G \times G']$-resolution of k. Recall that the tensor product of two resolutions C_* and D_* is the chain complex $C_* \otimes_k D_*$ where $(C_* \otimes_k D_*)_n = \oplus_{i+j=n} C_i \otimes_k D_j$ with boundary map defined by

$$\partial_n(x \otimes y) = \partial_i x \otimes y + (-1)^i x \otimes \partial_j y$$

for every $x \in C_i$ and $y \in D_j$. The fact that $P_* \otimes P'_* \to k$ is a resolution follows from the Künneth Theorem. Note that we identify $k \otimes_k k$ with k via the homomorphism $\mu : k \otimes_k k \to k$ defined by $\lambda \otimes \lambda' \to \lambda\lambda'$.

Lemma 96. *If $P_* \xrightarrow{\epsilon} k$ and $P'_* \xrightarrow{\epsilon'} k$ are kG-projective resolutions of k then the tensor product resolution*

$$P_* \otimes_k P_* \xrightarrow{\epsilon \otimes \epsilon'} k \otimes_k k \xrightarrow{\mu} k$$

is a kG-projective resolution of k when it is considered as a chain complex of kG-modules via the diagonal map $G \to G \times G$ defined by $g \to (g, g)$.

Proof. For each i, j the product $P_i \otimes_k P_j$ is a projective $k[G \times G]$-module. Since the diagonal map $G \to G \times G$ is injective, its restriction via the diagonal map is still projective. Hence $P_* \otimes P'_* \to k$ is a kG-projective resolution. The sequence $P_* \otimes_k P_* \to k$ is exact by Künneth theorem. □

Note that since any two kG-projective resolutions of k are chain homotopy equivalent, for a kG-projective resolution $P_* \to k$, there is a chain homotopy equivalence

$$\Delta : P_* \to P_* \otimes_k P_*$$

covering the identity map on k. Such a map is called a **diagonal approximation** if its compositions with the canonical maps

$$\pi_1 : P_* \otimes_k P_* \xrightarrow{\mathrm{id} \otimes \epsilon} P_* \otimes_k k \to P_*$$

$$\pi_2 : P_* \otimes_k P_* \xrightarrow{\epsilon \otimes \mathrm{id}} k \otimes_k P_* \to P_*$$

are equal to the identity map id_{P_*}. Diagonal approximations are important for computations when we are calculating cup products. For standard resolution we have the following well-known diagonal approximation which is a special case of Alexandre-Whitney diagonal approximation.

Lemma 97. *Let $F_* \to k$ be the standard kG-free resolution for k. Then the chain map $\Delta : F_* \to F_* \otimes_k F_*$ defined by*

$$\Delta(g_0, \dots, g_n) = \sum_{p=0}^{n} (g_0, \dots, g_p) \otimes (g_p, \dots, g_n)$$

is a diagonal approximation.

Proof. By direct calculation one can show that Δ is a chain map. It is also easy to check that its composition with π_1 and π_2 are equal to the identity map id_{P_*}. □

Now we will define the cup product for group cohomology. Let $P_* \to k$ be a kG-projective resolution of k and $\Delta : P_* \to P_* \otimes P_*$ be a diagonal approximation covering the identity map on k. For kG-modules M and N, consider the composition

$$\mathrm{Hom}_{kG}(P_*, M) \otimes_k \mathrm{Hom}_{kG}(P_*, N) \xrightarrow{\psi} \mathrm{Hom}_{kG}(P_* \otimes_k P_*, M \otimes_k N) \xrightarrow{\Delta^*} \mathrm{Hom}_{kG}(P_*, M \otimes_k N)$$

where the first map ψ is defined by

$$\psi(f_1 \otimes f_2)(x_1 \otimes x_2) = (-1)^{\deg f_2 \cdot \deg x_1} f_1(x_1) \otimes f_2(x_2)$$

and the second map is defined by $\Delta^*(f)(x) = f(\Delta(x))$. These chain maps induces a homomorphism of cohomology modules

$$H^n(\Delta^* \circ \psi) : H^n(\mathrm{Hom}_{kG}(P_*, M) \otimes_k \mathrm{Hom}_{kG}(P_*, N)) \to H^n(\mathrm{Hom}_{kG}(P_*, M \otimes_k N)).$$

Recall that by standard properties of tensor product of chain complexes, we have an homomorphism

$$\theta : H^p(\mathrm{Hom}_{kG}(P_*, M)) \otimes_k H^q(\mathrm{Hom}_{kG}(P_*, N)) \to H^{p+q}(\mathrm{Hom}_{kG}(P_*, M) \otimes_k \mathrm{Hom}_{kG}(P_*, N))$$

defined by $[f_1] \otimes [f_2] \to [f_1 \otimes f_2]$.

Definition 98. For every $p, q \geq 0$, the **cup product**

$$\cup : H^p(G; M) \otimes_k H^q(G; N) \to H^{p+q}(G, M \otimes_k N)$$

is defined to be the composition $\theta \circ H^n(\Delta^* \circ \psi)$. To shorten the notation we write uv for the cup product $u \cup v$.

The cup product satisfies many nice properties. For example it is natural with respect to homomorphisms $M \to M'$ and $N \to N'$ between the coefficient modules and with respect to group homomorphisms $f : G \to H$. The cup product also commutes with connecting homomorphisms induced by the short exact sequences of kG-modules. We also have

Proposition 99.

1. *For every $u_i \in H^{n_i}(G; M_i)$ for $i = 1, 2, 3$, we have*

$$(u_1 u_2) u_3 = u_1 (u_2 u_3)$$

in $H^{n_1+n_2+n_3}(G; M_1 \otimes_k M_2 \otimes_k M_3)$.

2. *Let M, N be kG-modules and $t : N \otimes_k M \to M \otimes_k N$ be the kG-module homomorphism defined by $t(m \otimes n) = n \otimes m$. For any $u \in H^p(G; M)$ and $v \in H^q(G; N)$, we have*

$$uv = (-1)^{pq} t_*(vu)$$

in $H^{p+q}(G; M \otimes_k N)$.

There is also a Frobenius reciprocity formula for the transfer map.

Proposition 100. *Let $H \leq G$ be a subgroup of G with finite index. Then for any $u \in H^p(G; M)$ and $v \in H^q(H; N)$, we have*

$$\mathrm{Tr}_H^G(\mathrm{Res}_H^G(u) \cdot v) = u \cdot \mathrm{Tr}_H^G v.$$

Proof. Let $u = [f_1]$ where $f_1 \in \mathrm{Hom}_{kG}(F_p, M)$ and $v = [f_2]$ where $f_2 \in \mathrm{Hom}_{kH}(\mathrm{Res}_H^G F_q, N)$. Then for $x \otimes y \in F_p \otimes_k F_q$, we have

$$\begin{aligned}
\mathrm{Tr}_H^G(\mathrm{Res}_H^G f_1 \otimes f_2)(x \otimes y) &= \sum_{gH \in G/H} (g \cdot (f_1 \otimes f_2)(g^{-1}x \otimes g^{-1}y)) \\
&= \sum_{gH \in G/H} (-1)^{p+q} g f_1(g^{-1}x) \otimes g f_2(g^{-1}y) \\
&= \sum_{gH \in G/H} (-1)^{p+q} f_1(x) \otimes g f_2(g^{-1}y) \\
&= (f_1 \otimes \mathrm{Tr}_H^G f_2)(x \otimes y).
\end{aligned}$$

This proves the desired equality once we apply Δ^* and pass to the cohomology. □

If both M and N are equal to the trivial module we can combine the cup product with the multiplication map $k \otimes_k k \to k$ and obtain a product

$$H^p(G; k) \otimes_k H^q(G; k) \to H^{p+q}(G; k)$$

for $p, q \geq 0$. Note that this induces a product

$$H^*(G; k) \otimes_k H^*(G; k) \to H^*(G; k)$$

on the graded k-module $H^*(G; k) = \sum_{i \geq 0} H^i(G; k)$.

Proposition 101. *For every group G and commutative ring k, the cup product defined a graded ring structure on $H^*(G; k)$. Moreover the cup product is graded commutative, i.e. for every $u \in H^p(G; k)$ and $v \in H^q(G; k)$ we have $uv = (-1)^{p+q} vu$.*

Proof. We proved the associativity and graded commutativity in Proposition 99. Note that $H^0(G; k) \cong k^G = k$. The unit element is $1 \in k \cong H^0(G; k)$. It is easy to check that $u \cdot 1 = 1 \cdot u = u$ for every $u \in H^p(G; k)$. □

For cyclic groups for calculating the group cohomology it is more practical to use the periodic resolution. So for calculating cup product for cyclic groups it is useful to have a diagonal approximation for the periodic resolution for cyclic groups.

Proposition 102 (pg 108 in [2])**.** *Let $G = \langle g \mid g^p = 1 \rangle$ be a finite cyclic group of order p. Let $F_* \to k$ be the periodic free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, where $F_n = \mathbb{Z}G$ for all $n \geq 0$. Then the map $\Delta : F_* \to F_* \otimes_k F_*$ defined by*

$$\Delta_{p,q}(1) = \begin{cases} 1 \otimes 1 & \text{if } p \text{ is even} \\ 1 \otimes g & \text{if } p \text{ is odd}, q \text{ is even} \\ \sum_{0 \leq i < j \leq p-1} g^i \otimes g^j & \text{if } p \text{ is odd}, q \text{ is odd} \end{cases}$$

is a diagonal approximation.

Exercise 103. Write a proof for the above proposition.

Exercise 104. Using the diagonal approximation given in the proposition calculate the ring structure of $H^*(G;k)$ when G is a cyclic group of order n.

6 More Homological Algebra

6.1 Snake Lemma and Five Lemma

One of the most important tools in homological algebra is the Snake lemma.

Lemma 105 (Snake Lemma)**.** *Given a diagram of R-modules*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

there is an R-module homomorphism $\partial : \ker\gamma \to \operatorname{coker}\alpha$, called connecting homomorphism, such that the sequence

$$0 \to \ker f \xrightarrow{i} \ker\alpha \xrightarrow{f} \ker\beta \xrightarrow{g} \ker\gamma \xrightarrow{\partial} \operatorname{coker}\alpha \xrightarrow{f'} \operatorname{coker}\beta \xrightarrow{g'} \operatorname{coker}\gamma \xrightarrow{p} \operatorname{coker} g' \to 0$$

is exact, where i is an inclusion map and p is the projection map induced by the diagram.

Proof. We first describe how the connecting homomorphism is defined. Let $c \in \ker\gamma$. Since g is an epimorphism, there is a $b \in B$ such that $g(b) = c$. We have $g'\beta(b) = \gamma g(b) = \gamma(c) = 0$. This gives

$$\beta(b) \in \ker g' = \operatorname{im} f'.$$

Hence there is an $a \in A'$ such that $f'(a) = \beta(b)$. We define $\partial : \ker\gamma \to \operatorname{coker}\alpha$ to be the R-module homomorphism which sends $c \in \ker\gamma$ to $\overline{a} = a + \operatorname{im}\alpha$ in $\operatorname{coker}\alpha$.

This homomorphism is well-defined because if we choose another element $b' \in B$ such that $g(b') = c$, then $g(b' - b) = 0$. This means $b' - b \in \ker g = \operatorname{im} f$. So there is a (unique) $u \in A$ such that $b' - b = f(u)$. If $a' \in A$ such that $f(a') = \beta(b')$, then

$$f'\alpha(u) = \beta f(u) = \beta(b' - b) = \beta(b') - \beta(b) = f(a') - f(a) = f(a' - a).$$

This gives $\alpha(u) = a' - a$. Hence $\overline{a'} = \overline{a}$.

If $a \in \ker f$, then $f'\alpha(a) = \beta f(a) = 0$. Since f' is a monomorphism, $\alpha(a) = 0$. So $\ker f \subseteq \ker\alpha$. So the first map $i : \ker f \to \ker\alpha$ is inclusion map and the sequence is exact at both $\ker f$ and $\ker\alpha$.

It is clear that the composition $gf : \ker\alpha \to \ker\gamma$ is zero. If $b \in \ker\beta$ is such that $g(b) = 0$, then there is an $a \in A$ such that $f(a) = b$. We have $f'\alpha(a) = \beta f(a) = \beta(b) = 0$. So $\alpha(a) = 0$. Hence $a \in \ker\alpha$ and $f(a) = b$. This shows that the sequence is exact at $\ker\beta$.

If $c \in \ker\gamma$ such that $c = g(b)$ for some $b \in \ker\beta$. Then $\beta(b) = 0$, so $a \in A'$ such that $f'(a) = \beta(b)$ can be taken as 0. So $\partial(c) = 0$ in $\operatorname{coker}\alpha$. Hence $\partial g = 0$. Let $c \in \mathit{coker}\gamma$ such that $\partial(c) = 0$. Then there is a $b \in B$ and $a \in A'$ such that $g(b) = c$ and $f'(a) = \beta(b)$. Moreover since $[a] = 0$ in $\operatorname{coker}\alpha$, there is a $u \in A$ such that $\alpha(u) = a$. This gives $\beta f(u) = f'(a) = \beta(b)$. Hence $b' = b - f(u) \ker\beta$ and $g(b') = c$. Hence the sequence is exact at $\ker\gamma$.

For $c \in \ker\gamma$, we have $\partial(c) = \overline{a}$ such that there is a $b \in B$ satisfying $g(b) = c$ and $f'(a) = \beta(b)$. Then $f'(a) \in \operatorname{im}\beta$, hence $f'(\overline{a}) = 0$ in $\operatorname{coker}\beta$. This gives that $f'\partial = 0$. Let $\overline{a} \in \operatorname{coker}\alpha$ such that $f'(\overline{a}) = 0$ in $\operatorname{coker}\beta$. This means there is a $b \in B$ such that $f'(a) = \beta(b)$. We have $\gamma g(b) = g'\beta(b) = g'f'(a) = 0$. So, $c = g(b) \ker\gamma$, and $\partial(c) = \overline{a}$. Hence the sequence is exact at $\operatorname{coker}\alpha$.

The composition $g'f' : \operatorname{coker}\alpha \to \operatorname{coker}\gamma$ is zero since $g'f' : A' \to C'$ is the zero map. Take $\overline{b} \in \operatorname{coker}\beta$ such that $\overline{g'} = 0$ in $\operatorname{coker}\gamma$. This means $g'(b) = \gamma(c)$ for some $c \in C$. Since g is an epimorphism, there is $y \in B$ such that $g(y) = c$. We have $g'\beta(y) = \gamma g(y) = \gamma(c) = g'(b)$. So there is an $a \in A'$ such that $f'(a) = b - \beta(y)$. This gives $\overline{b} = \overline{f'(a)}$, hence $\overline{b} \in \operatorname{im} f'$. We conclude that the sequence is exact at $\operatorname{coker}\beta$.

Let $c \in \operatorname{im}\gamma$. Then $c = \gamma(x)$ for some $x \in C$. Since g is surjective, there is an element $b \in B$ such that $g(b) = x$. We have $g'\beta(b) = \gamma g(b) = \gamma(x) = c$. So, $c \in \operatorname{im} g'$. This shows that $\operatorname{im}\gamma \subseteq \operatorname{im} g'$. This induces a surjective homomorphism $p : \operatorname{coker}\gamma \to \operatorname{coker} g'$. The kernel of p is equal to the quotient module $\operatorname{im} g'/\operatorname{im}\gamma$. From this it is clear that the composition pg' is equal to zero and the sequence is exact at $\operatorname{coker}\gamma$. □

The connecting homomorphism in the Snake lemma is natural in a suitable sense.

Lemma 106. *The connecting homomorphism constructed in 105 is natural with respect to a morphism of such diagrams.*

Proof. Suppose that there are two diagrams

$$\begin{array}{ccccccc} & & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle \gamma_i} & & \\ 0 & \longrightarrow & A'_i & \xrightarrow{f'_i} & B'_i & \xrightarrow{g'_i} & C'_i & & \end{array}$$

for $i = 1, 2$ with homomorphisms $j : A_1 \to A_2$, $k : B_1 \to B_2$, $l : C_1 \to C_2$, and $j' : A'_1 \to A'_2$, $k' : B'_1 \to B'_2$, $l' : C'_1 \to C'_2$ that commutes with all the maps in the diagrams for $i = 1, 2$. We claim that if $\partial_i : C_i \to A'_i$ for $i = 1, 2$ is the connecting homomorphism for each diagram, then the following diagram commutes:

$$\begin{array}{ccc} \ker\gamma_1 & \xrightarrow{\partial_1} & \operatorname{coker}\alpha_1 \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \overline{j'}} \\ \ker\gamma_2 & \xrightarrow{\partial_2} & \operatorname{coker}\alpha_2 \end{array}$$

where $l : \ker\gamma_1 \to \ker\gamma_2$ and $\overline{j'} : \operatorname{coker}\alpha_1 \to \operatorname{coker}\alpha_2$ are the maps induced by the maps $l : C_1 \to C_2$ and $j' : A'_1 \to A'_2$.

Let $c \in \ker\gamma_1$ and $b \in B_1$ and $a \in A'_1$ are such that $g_1(b) = c$ and $f'_1(a) = \beta_1(b)$. This gives that $\partial_1(c) = \overline{a}$. Note that $g_2k(b) = lg_1(b) = l(c)$, and $f'_2j'(a) = k'f'_1(a) = k'\beta_1(b) = \beta_2k(b)$. From these we obtain that $\partial_2 l(c) = \overline{j'(a)} = \overline{j'}(\overline{a}) = \overline{j'}\partial_1(c)$. Hence the diagram above commutes. □

Another important lemma in homological algebra is the Five Lemma.

Lemma 107 (Five Lemma)**.** *Consider the following commutative diagram*

$$
\begin{array}{ccccccccc}
A_1 & \xrightarrow{f} & A_2 & \xrightarrow{g} & A_3 & \xrightarrow{h} & A_4 & \xrightarrow{k} & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{f'} & B_2 & \xrightarrow{g'} & B_3 & \xrightarrow{h'} & B_4 & \xrightarrow{k'} & B_5
\end{array}
$$

where both of the horizontal sequences are exact.

1. *If f_1 is surjective, and f_2, f_4 are injective, then f_3 is injective.*
2. *If f_5 is injective, and if f_2, f_4 are surjective, then f_3 is surjective.*

As a consequence, if f_1, f_2, f_4, f_5 are all isomorphisms, then f_3 is an isomorphism.

Proof. (1) Assume that $x \in A_3$ such that $f_3(x) = 0$. Then $f_4h(x) = h'f_3(x) = 0$. Since f_4 is injective, we obtain $h(x) = 0$. By the exactness of the top horizontal sequence, there is an element $y \in A_2$ such that $g(y) = x$. We have $g'f_2(y) = f_3g(y) = f_3(x) = 0$. Thus $f_2(y) \in \ker g'$. By the exactness of the bottom sequence, there is an element $z \in B_1$ such that $f'(z) = f_2(y)$. Since f_1 is surjective, there is $w \in A_1$ such that $f_1(w) = z$. We have $f_2f(w) = f'f_1(w) = f'(z) = f_2(y)$. Since f_2 is injective, this gives $y = f(w)$. Thus $x = g(y) = gf(w) = 0$. Hence f_3 is injective.

(2) Take $x \in B_3$. Since f_4 is surjective, there is $y \in A_4$ such that $f_4(y) = h'(x)$. We have $f_5k(y) = k'f_4(y) = k'h'(x) = 0$. Since f_5 is injective, $k(y) = 0$. Since the top horizontal sequence is exact, there is an element $z \in A_3$ such that $h(z) = y$. We have $h'f_3(z) = f_4h(z) = f_4(y) = h'(x)$. Thus $x - f_3(z)$ is in $\ker h'$. By the exactness of the bottom sequence, there is an element $w \in B_2$ such that $g'(w) = x - f_3(z)$. Since f_2 is surjective, there is an element $u \in A_2$ such that $f_2(u) = w$. We have

$$f_3(z + g(u)) = f_3(z) + f_3g(u) = f_3(z) + g'f_2(u) = f_3(z) + g'(w) = f_3(z) + x - f_3(z) = x.$$

Hence $x \in \operatorname{im} f_3$. We conclude that f_3 is surjective. $\square$

Lemma 108 (3×3-Lemma)**.** *Consider the following commuting diagram of R-modules:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1' & \longrightarrow & B_2' & \longrightarrow & C_2' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1'' & \longrightarrow & B_2'' & \longrightarrow & C_2'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

1. *If all the vertical lines are exact sequences and either top two horizontal lines or bottom two horizontal lines are exact, then the third horizontal line is exact.*

2. *If all the horizontal lines are exact sequences and either left two vertical lines or right two vertical lines are exact, then the third vertical line is exact.*

Exercise 109. Prove the 3×3-lemma using the Snake lemma.

Exercise 110. For any homomorphisms $\alpha : A \to B$ and $\beta : B \to C$, show that there is an exact sequence

$$0 \to \ker \alpha \to \ker \beta\alpha \to \ker \beta \to \operatorname{coker} \alpha \to \operatorname{coker} \beta\alpha \to \operatorname{coker} \beta \to 0.$$

6.2 Long Exact Sequences for Chain Complexes

Definition 111. A chain complex B_* is a **subcomplex** of C_* if for each $n \in \mathbb{Z}$, B_n is a submodule of C_n and the inclusion maps $i_n : B_n \hookrightarrow C_n$ gives a chain map $i_* : B_* \to C_*$.

Exercise 112. Consider the following chain complexes:

$$B_* : \cdots \to 2.\mathbb{Z}_8 \xrightarrow{0} 2.\mathbb{Z}_8 \xrightarrow{0} \ldots \xrightarrow{0} 2.\mathbb{Z}_8 \xrightarrow{0} 2.\mathbb{Z}_8 \to \cdots$$

$$C_* : \cdots \to \mathbb{Z}_8 \xrightarrow{\times 4} \mathbb{Z}_8 \xrightarrow{\times 4} \ldots \xrightarrow{\times 4} \mathbb{Z}_8 \xrightarrow{\times 4} \mathbb{Z}_8 \to \cdots$$

Here B_* is a subcomplex of C_*. Note that $H_n(B_*) = 2\mathbb{Z}_8 \cong \mathbb{Z}/4$ for all $n \in \mathbb{Z}$ and $H_n(C_*) \cong \mathbb{Z}_2$ for $n \in \mathbb{Z}$. So the inclusion map $i_* : B_* \to C_*$ of a subcomplex does not induce an injective R-module homomorphism on homology.

To understand the relationship between homology groups of B_* and C_* we need to consider the homology groups of the quotient complex.

Definition 113. If B_* is a subcomplex of C_* then we can define the quotient complex C_*/B_* to be the chain complex

$$\cdots \to C_{n+1}/B_{n+1} \xrightarrow{d_{n+1}} C_n/B_n \xrightarrow{d_n} C_{n-1}/B_{n-1} \to \ldots$$

where $d_n(x + B_n) = d_n^C(x) + B_{n-1}$ for all $n \in \mathbb{Z}$. Note that this defines a well-defined R-module homomorphism because $d_n^C(B_n) \subseteq B_{n-1}$.

Note that if B_* is a subcomplex of C_*, then there is a chain map $q_* : C_* \to C_*/B_*$ which is defined on each group as the R-module homomorphism $q_n : C_n \to C_n/B_n$ which is define by $q_n(x) = x + B_n$.

Definition 114. Given a chain map $f_* : C_* \to D_*$, the kernel chain map $\ker f$ is the chain complex with chain groups $(\ker f_*)_n = \ker f_n$ with boundary maps induced by the boundary maps of C_*. The image of f_* is the subcomplex of D_* with chain groups $(\operatorname{im} f_*)_n = \operatorname{im} f_n$. The cokernel of f_* is defined to be the quotient complex $D_*/\operatorname{im} f_*$.

For chain complexes, the monomorphisms and epimorphisms are defined in a similar way to R-modules. A chain map $f_* : C_* \to D_*$ is a **epimorphism** if $\operatorname{coker} f_* = 0$, and a **monomorphism** if $\ker f_* = 0$.

Definition 115. A sequence of chain complexes

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

is a **short exact sequence** if for every $n \in \mathbb{Z}$, the sequence

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is exact.

The main result of this section is the following:

Theorem 116. *For every short exact sequence of chain complexes*

$$0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$$

there is a homomorphism $\partial_n : H_n(C_*) \to H_{n-1}(A_*)$, called the **connecting homomorphism***, such that the sequence*

$$\cdots \xrightarrow{g_*} H_{n+1}(C_*) \xrightarrow{\partial_{n+1}} H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \xrightarrow{f_*} \cdots$$

is exact. Moreover the connecting homomorphism is natural with respect to the morphism of short exact sequences.

Proof. Fix an $n \in \mathbb{Z}$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\ 0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \end{array}$$

By Snake Lemma, there is an exact sequence

$$0 \to Z_n(A_*) \to Z_n(B_*) \to Z_n(C_*) \xrightarrow{\partial}$$
$$A_{n-1}/B_{n-1}(A_*) \to B_{n-1}/B_{n-1}(B_*) \to C_{n-1}/B_{n-1}(C_*) \to 0.$$

Taking the first three terms of this sequence with a dimension shift by -1, and the last three terms of the sequence with a dimension shift by $+1$, we obtain a diagram of the form:

$$\begin{array}{ccccccccc} & & A_n/B_n(A_*) & \longrightarrow & B_n/B_n(B_*) & \longrightarrow & C_n/B_n(C_*) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\ 0 & \longrightarrow & Z_{n-1}(A_*) & \longrightarrow & Z_{n-1}(B_*) & \longrightarrow & Z_{n-1}(C_*) & & \end{array}$$

Applying the Snake lemma to this diagram, we obtain an exact sequence

$$H_n(A_*) \to H_n(B_*) \to H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \to H_{n-1}(B_*) \to H_{n-1}(C_*).$$

Since we have such an exact sequence for every $n \in \mathbb{Z}$, the long exact homology sequence given in the statement of the theorem is exact. The naturality of the connecting homomorphism follows from Lemma 106. $\square$

Example 117. Let C_* be a chain complex of projective R-modules. Given a short exact sequence of R-modules

$$0 \to N \xrightarrow{i} M \xrightarrow{\pi} L \to 0,$$

when apply the Hom-functor $\mathrm{Hom}_R(C_*, -)$ to this exact sequence. Since C_n is projective for all $n \in \mathbb{Z}$, this gives a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}_R(C_*, N) \to \mathrm{Hom}_R(C_*, M) \to \mathrm{Hom}_R(C_*, L) \to 0.$$

By Theorem 116 we obtain a long exact sequence

$$\cdots \xrightarrow{\pi_*} H^{n-1}(C_*; L) \xrightarrow{\delta^{n-1}} H^n(C_*, N) \xrightarrow{i_*} H^n(C_*; M) \xrightarrow{\pi_*} H^n(C_*; L) \xrightarrow{\delta^n} H^{n+1}(C_*; N) \xrightarrow{i_*} \cdots$$

Example 118. In the above example if we take $R = \mathbb{Z}G$ and let C_* to be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, then for every sequence of $\mathbb{Z}G$-modules

$$0 \to N \xrightarrow{i} M \xrightarrow{\pi} L \to 0$$

we obtain a long exact sequence for group cohomology

$$\cdots \xrightarrow{\pi_*} H^{n-1}(G; L) \xrightarrow{\delta^{n-1}} H^n(G; N) \xrightarrow{i_*} H^n(G; M) \xrightarrow{\pi_*} H^n(G; L) \xrightarrow{\delta^n} H^{n+1}(G; N) \xrightarrow{i_*} \cdots$$

Example 119. (Bockstein Homomorphism) In the first example above if we take C_* to be the singular chain complex of a topological space X or the simplicial chain complex of a simplicial complex X, then applying Theorem 116 to the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{m_2} \mathbb{Z}/2 \to 0,$$

we obtain a long exact sequence

$$\cdots \xrightarrow{m_2} H^{n-1}(X; \mathbb{Z}/2) \xrightarrow{\delta^{n-1}} H^n(Z; \mathbb{Z}) \xrightarrow{\times 2} H^n(X; \mathbb{Z}) \xrightarrow{m_2} H^n(X; \mathbb{Z}/2) \xrightarrow{\delta^n} H^{n+1}(X; \mathbb{Z}) \xrightarrow{\times 2} \cdots$$

In this case the connecting homomorphism $\delta^n : H^n(X; \mathbb{Z}/2) \to H^{n+1}(X; \mathbb{Z})$ is called the integral Bockstein homomorphism, denoted by β_0.

If we apply Theorem 116 to the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

then the connecting homomorphism $\delta^n : H^n(X; \mathbb{Z}/2) \to H^{n+1}(X; \mathbb{Z}/2)$ for this sequence is called the Bockstein homomorphism, denoted by β. Note that by the naturality of connecting homomorphisms we have $\beta = m_2 \beta_0$.

Exercise 120. Write the details of the argument to prove that $\beta = m_2 \beta_0$.

As a corollary of Theorem 116, we have the following:

Proposition 121. *Consider the commuting diagram of chain complexes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f_*} & B_* & \xrightarrow{g_*} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{f'_*} & B'_* & \xrightarrow{g'_*} & C'_* & \longrightarrow & 0.
\end{array}$$

If any two of the three chain maps α_*, β_*, and γ_* induces isomorphism on all homology modules, then the third one also does.

Proof. By Snake lemma there is a commuting diagram of long exact homology sequences:

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & H_{n+1}(C_*) & \longrightarrow & H_n(A_*) & \longrightarrow & H_n(B_*) & \longrightarrow & H_n(C_*) & \longrightarrow & H_{n-1}(A_*) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & \\
\cdots \longrightarrow & H_{n+1}(C_*) & \longrightarrow & H_n(A_*) & \longrightarrow & H_n(B_*) & \longrightarrow & H_n(C_*) & \longrightarrow & H_{n-1}(A_*) & \longrightarrow \cdots
\end{array}
$$

If α_* and γ_* both induces isomorphism on homology, then by Five lemma, β_* also induces isomorphism on homology. The same argument also proves the other two possibilities. □

Exercise 122. Consider the following commutative diagram of chain complexes with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{j} & B'_* & \xrightarrow{q} & C'_* & \longrightarrow & 0
\end{array}
$$

Prove that if $h_n : H_n(C_*) \to H_n(C'_*)$ is an isomorphism for all $n \in \mathbb{Z}$, then there is an exact sequence

$$\cdots \longrightarrow H_n(A_*) \xrightarrow{(f_n,i_n)} H_n(A'_*) \oplus H_n(B_*) \xrightarrow{j_n - g_n} H_n(B'_*) \xrightarrow{\partial_n h_n^{-1} q_n} H_{n-1}(A_*) \xrightarrow{(f_{n-1},i_{n-1})} \cdots$$

6.3 Universal Coefficient Theorems and Künneth Theorem

When R is a commutative ring which is a principle ideal domain (PID), there is a theorem which explains the relationship between $H_n(C_* \otimes_R M)$ and $H_n(C_*) \otimes_R M$.

Theorem 123. *Suppose that R is a PID, C_* is a chain complex of free R-modules C_*, and M is a R-module M. Then for every $n \in \mathbb{Z}$, there is a short exact sequence of abelian groups*

$$0 \to H_n(C_*) \otimes_R M \xrightarrow{\varphi} H_n(C_* \otimes_R M) \xrightarrow{\psi} \operatorname{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

where $\varphi([z] \otimes m) = [z \otimes m]$ for every $[z] \in H_n(C_*)$ and $m \in M$. The sequence is natural with respect to C_* and M, and it splits but the splitting is not natural.

Proof. Let

$$C_* : \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of free R-modules. For each $n \in \mathbb{Z}$, let $Z_n = \ker \partial_n$ and $B_n = \operatorname{im} \partial_{n+1}$. Since R is a PID and C_n is free R-module, both Z_n and B_n are free R-modules. By definition of cohomology groups, for each $n \in \mathbb{Z}$, there is a short exact sequence of R-modules

$$0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{q_n} H_n(C_*) \to 0.$$

Taking the tensor product with M over R gives a sequence

$$0 \to \operatorname{Tor}_1^R(H_{n-1}(C_*), M) \to B_n \otimes_R M \xrightarrow{j_n \otimes id_M} Z_n \otimes_R M \to H_n(C_*) \otimes_R M \to 0.$$

From this we obtain that

$$\ker(j_n \otimes \mathrm{id}_M) \cong \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \ \text{ and } \ \mathrm{coker}\,(j_n \otimes \mathrm{id}_M) \cong H_n(C_*) \otimes_R M.$$

For each $n \in \mathbb{Z}$, there is a sequence of R-modules

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{d_n} B_{n-1} \to 0.$$

Let Z_* be the chain complex with zero differentials and such that $(Z_*)_n = Z_n$. Similarly let ΣB_* be the chain complex with zero differentials and such that $(\Sigma B_*)_n = B_{n-1}$. We can think of ΣB_* as the chain complex B_* which is shifted towards the positive dimension by one degree. The short exact sequence above gives a short exact sequence of chain complexes

$$0 \to Z_* \xrightarrow{i_*} C_* \xrightarrow{d_*} \Sigma B_* \to 0.$$

Since B_* is a cochain complex of free R-modules, taking tensor product with M over R gives an exact sequence of chain complexes

$$0 \to Z_* \otimes_R M \xrightarrow{i_* \otimes \mathrm{id}_M} C_* \otimes_R M \xrightarrow{d_* \otimes \mathrm{id}_M} \Sigma B_* \otimes_R M \to 0.$$

By Theorem 116, there is a long exact sequence of homology modules

$$\cdots \xrightarrow{d_* \otimes \mathrm{id}_M} H_{n+1}(\Sigma B_* \otimes_R M) \xrightarrow{\partial_{n+1}} H_n(Z_* \otimes_R M) \xrightarrow{i_* \otimes \mathrm{id}_M} H_n(C_* \otimes_R M)$$

$$\xrightarrow{d_* \otimes \mathrm{id}_M} H_n(\Sigma B_* \otimes_R M) \xrightarrow{\partial_n} H_{n-1}(Z_* \otimes_R M) \xrightarrow{i_* \otimes \mathrm{id}_M} \cdots$$

where ∂_n denotes the connecting homomorphism of the long exact sequence.

Note that since the differentials of Z_* and ΣB_* are zero, we have $H_m(Z_* \otimes M) \cong Z_* \otimes M$ and $H_{n+1}(\Sigma B_* \otimes_R M) \cong B_n \otimes_R M$. We claim that ∂_{n+1} is equal to the map $j_n \otimes_R M : B_n \otimes_R M \to Z_n \otimes_R M$. To see this take $b \otimes m \in B_n \otimes_R M$. Let $x \in C_{n+1}$ such that $d_n x = b$. Applying $d_n \otimes \mathrm{id}_M$ to $x \times m$ gives an element in $C_{n-1} \otimes_R M$ such that $(d_{n-1} \otimes \mathrm{id}_m)(d_n(x) \otimes m) = 0$. So, $d_n(x) \otimes m \in i_* \otimes \mathrm{id}_M$. Since i_* is just the inclusion map, by the definition of connecting homomorphism we obtain $\partial_{n+1}(b \otimes m) = d_n(x) \otimes m$. Since $d_n(x) = b$, we obtain that $\partial_{n+1} = j_n \otimes \mathrm{id}_M$.

The long exact sequence above gives a short exact sequence

$$0 \to \mathrm{coker}\,(\partial_{n+1}) \to H_n(C_* \otimes_R M) \to \ker \partial_n \to 0.$$

Since $\partial_{n+1} = j_n \otimes \mathrm{id}_M$, by our earlier calculations for the kernel and cokernel of these homomorphisms, we obtain a short exact sequence

$$0 \to H_n(C_*) \otimes_R M \xrightarrow{\varphi} H_n(C_* \otimes_R M) \xrightarrow{\psi} \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to 0.$$

It is easy to see from the arguments above that the first map is given by $[z] \otimes m \to [z \otimes m]$.

To see that why the sequence splits, note that for each $n \in \mathbb{Z}$, the sequence

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{d_n} B_{n-1} \to 0$$

splits because B_{n-1} is R-free. This means that there is a homomorphism $t_n : C_n \to Z_n$ such that $t_n \circ i_n = \mathrm{id}_{Z_n}$. Given $u = [\sum_k x_k \otimes m_k] \in H_n(C_* \otimes_R M)$ with $x_k \in C_n$ and $m_k \in M$, we see that $\sum_k t_n(x_k) \otimes m_k \in Z_n \otimes_R M$. A splitting for φ can be defined by $t'(u) = \sum_k [t_n(x_k)] \otimes m_k$. Note that

$$(t' \circ \varphi)([z] \otimes m) = t'([i_n(z) \otimes m]) = [t_n i_n(z)] \otimes m = [z] \otimes m.$$

□

There is also a Universal Coefficient Theorem for cohomology.

Theorem 124. *Let R be a PID ring. If C_* is a chain complex of free R-modules and M is an R-module, then for every $n \in \mathbb{Z}$, there is a short exact sequence of abelian groups*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), M) \xrightarrow{\psi} H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\varphi} \mathrm{Hom}_R(H_n(C_*), M) \to 0$$

where $\varphi([f])([z]) = f(z)$ for every $f \in \mathrm{Hom}_R(C_n, M)$ and $z \in \ker d_n$. The sequence is natural with respect to C_* and M, and it splits but splitting is not natural.

Proof. The argument is similar to the argument given for the Universal Coefficient Theorem for Homology. Let C_* be a cahin complex of free R-modules and M be an R-module. Consider the sequence

$$0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{q_n} H_n(C_*) \to 0$$

where $Z_n = \ker d_n$ and $B_n = \mathrm{im}\, d_n$ for every $n \in \mathbb{Z}$. Since R is a PID, both Z_n and B_n are R-free. Applying the Hom-functor $\mathrm{Hom}_R(-; M)$, we obtain a sequence

$$0 \to \mathrm{Hom}_R(H_n(C_*), M) \to \mathrm{Hom}_R(B_n, M) \xrightarrow{j_n^*} \mathrm{Hom}_R(Z_n, M) \to \mathrm{Ext}_R(H_n(C_*), M) \to 0.$$

This gives that

$$\ker j_n^* \cong \mathrm{Hom}_R(H_n(C_*), M) \quad \text{and} \quad \mathrm{coker}\, j_n^* \cong \mathrm{Ext}_R(H_n(C_*), M).$$

Now consider the sequence of chain complexes

$$0 \to Z_* \xrightarrow{i_*} C_* \xrightarrow{d_*} \Sigma B_* \to 0$$

where Z_* and ΣB_* has zero boundary maps. Applying $\mathrm{Hom}_R(-, M)$ we get

$$0 \to \mathrm{Hom}_R(\Sigma B_*, M) \xrightarrow{d^*} \mathrm{Hom}_R(C_*, M) \xrightarrow{i^*} \mathrm{Hom}_R(Z_*, M) \to 0.$$

Note that these sequences are split exact sequences at each chain group (does not split as sequence of chain complexes). Consider the associated long exact sequence of cohomology groups

$$\cdots \to H^{n-1}(\mathrm{Hom}_R(Z_*, M)) \xrightarrow{\delta^{n-1}} H^n(\mathrm{Hom}_R(\Sigma B_*, M)) \xrightarrow{d^*} H^n(\mathrm{Hom}_R(C_*, M))$$

$$\xrightarrow{i^*} H^n(\mathrm{Hom}_R(Z_*, M)) \xrightarrow{\delta^n} H^{n+1}(\mathrm{Hom}_R(\Sigma B_*, M)) \xrightarrow{d^*} \cdots$$

Since the differentials on Z_* and ΣB_* are zero maps, we have $H^n(\mathrm{Hom}_R(Z_*, M)) \cong \mathrm{Hom}_R(Z_*, M)$ and $H^{n+1}(\mathrm{Hom}_R(\Sigma B_*, M)) \cong \mathrm{Hom}_R(B_n, M)$. By an argumant similar to the one given for homology, we can show that the connecting homomorphism δ^n coincides with the homomorphism $j_n^* = \mathrm{Hom}_R(Z_n, M) \to \mathrm{Hom}_R(B_n, M)$. The long exact sequence above gives a short exact sequence

$$0 \to \operatorname{coker} \delta^{n-1} \to H^n(\mathrm{Hom}_R(C_*, M)) \to \ker \delta^n \to 0.$$

By using our earlier observations for the kernel and cokernel of j_n^*, we obtain the short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), M) \xrightarrow{\psi} H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\varphi} \mathrm{Hom}_R(H_n(C_*), M) \to 0.$$

From the construction we have $\varphi([f])([z]) = f(z)$ for every $f \in \mathrm{Hom}_R(C_n, M)$ and $z \in \ker d_n$. The fact that the sequence splits can be proved using the fact that the sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits. □

There are two more Universal Coefficients theorems but these theorems require more conditions to holds.

Theorem 125 (Theorem 2.33 in [3])**.** *Let R be a PID. If M is a finitely generated R-module and C_* is a chain complex of free R-modules, then there is a split short exact sequence*

$$H^n(C_*) \otimes_R M \to H^n(C_* \otimes_R M) \to \mathrm{Tor}_1^R(H^{n+1}(C_*), M).$$

Proof. See [**?**, pg. 246] for a proof. □

We also have the following:

Theorem 126 (Theorem 2.36 in [3])**.** *Let R be a PID. Suppose that C_* is a chain complex of free R-modules such that $H_n(C_*)$ is finitely generated for each n. Then for every R-module M, there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H^{n+1}(C_*), M) \to H_n(C_* \otimes_R M) \to \mathrm{Hom}(H^n(C_*), M) \to 0$$

which splits (non-naturally).

We conclude this section with the statement of the Künneth's theorem.

Theorem 127. *Let R be a PID. Suppose that C_* and D_* are chain complexes of R-modules where C_n is a free R-module for every $n \in \mathbb{Z}$. Then there is a natural exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes_R H_j(D_*) \to H_n(C_* \otimes_R D_*) \to \bigoplus_{i+j=n} \mathrm{Tor}_1^R(H_i(C_*), H_{j-1}(D_*) \to 0$$

which splits (non-naturally).

Proof. See pg. 53 in [3] □

A Background on Algebra and Topology

A.1 Abelian Categories

In this section we follow the definitions from [4].

Definition 128. A category $\mathscr{A}$ is called an **Ab-category** if every hom-set $\mathrm{Hom}_{\mathscr{A}}(x, y)$ in $\mathscr{A}$ has the structure of an abelian group in such a way that composition distributes over addition, i.e. given $f \in \mathrm{Hom}_{\mathscr{A}}(x, y)$, $g, g' \in \mathrm{Hom}_{\mathscr{A}}(y, z)$, and $h \in \mathrm{Hom}_{\mathscr{A}}(z, t)$, we have $h(g + g')f = hgf + hg'f$.

Example 129. The category of R-modules for every ring R is an *Ab*-category. For every $f, g \in \mathrm{Hom}_{R-Mod}(M, N)$, we can define $f + g$ to the the R-module homomorphism defined by $(f + g)(m) = f(m) + g(m)$ for every $m \in M$. It is easy to see that this addition of two homomorphisms distributes over the composition of homomorphisms.

Definition 130. An **additive functor** $F : \mathscr{A} \to \mathscr{B}$ between two Ab-categories is a functor such that for each $A, A' \in Ob(\mathscr{A})$, the map

$$\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A'))$$

that sends $f : A \to A'$ to $F(f) : F(A) \to F(A')$ is an abelian group homomorphism.

Definition 131. An **initial object** (respectively, a **terminal object**) in a category $\mathcal{C}$ is an object x such that for every $y \in \mathrm{Ob}(\mathcal{C})$, there is exactly one morphism from x to y (respectively, from y to x). An object $x \in \mathcal{C}$ is called a **zero object** if it is both initial and terminal object in $\mathscr{C}$. Zero object is unique up to unique isomorphism.

Example 132. In the category of R-modules the zero object is the zero module.

Recall that for a given set of objects $\{y_i : i \in I\}$ in a category $\mathcal{C}$ the **product** in $\mathcal{C}$ is an object $\prod_{i\in I} y_i \in \mathcal{C}$ together with maps $\pi_j : \prod_{i\in I} y_i \to y_j$ such that for every $x \in \mathcal{C}$ and for every family of morphisms $\{\alpha_i : x \to y_i\}_{i\in I}$, there is a unique morphism $\alpha : x \to \prod_{i\in I} y_i$ such that $\pi_j \alpha = \alpha_j$ for every $j \in I$. Products may not exist in a category. When they exist, there maybe more than one product but they are unique up to unique isomorphism. Using the dual picture we define the coproduct $\coprod_{i\in I} y_i$ as an object in $\mathcal{C}$ together with morphisms $i_j : y_j \to \coprod_{i\in I} y_i$ such that for every family of morphisms $\{\alpha_i : y_i \to z\}_{i\in I}$, there is a unique morphism $\alpha : \coprod_{i\in I} y_i \to z$ such that $\alpha i_j = \alpha_j$ for all $j \in I$.

Definition 133. An *Ab*-category $\mathscr{A}$ is called an **additive category** if it has a zero object and has a product $A \times B$ for every pair A, B of objects in $\mathscr{A}$.

Example 134. The category of R-modules is an additive category. The product is defined as the direct sum of two modules $M \oplus N$ which is the product of abelian groups $M \times N$ as an abelian group where the action of the ring R is given by

$$r(m, n) = (rm, rn) \text{ for all } r \in R, m \in M, n \in N.$$

Exercise 135. Show that in an additive category $\mathscr{A}$, all finite products and coproducts exists and they are isomorphic (i.e. a product is also a coproduct)

Exercise 136. Show that in an additive category $\mathscr{A}$, the zero morphisms $B \to 0 \to C$ is equal to $0 \in \mathrm{Hom}_{\mathscr{C}}(B, C)$.

Before we define abelian categories, we recall some definitions from category theory.

Definition 137.

1. A morphism $f : x \to y$ in a category $\mathcal{C}$ is **monic** if for every distinct morphisms $g_1, g_2 : z \to x$ in $\mathcal{C}$ we have $fg_1 \neq fg_2$ (or we can say $fg_1 = fg_2$ implies $g_1 = g_2$).
2. A morphism $f : x \to y$ in a category $\mathcal{C}$ is **epi** if for every distinct morphisms $g_1, g_2 : y \to t$ in $\mathcal{C}$ we have $g_1 f \neq g_2 f$.
3. If $\mathcal{C}$ is a category with zero object, the morphism $x \to 0 \to y$ is called the **zero morphism** in $\mathrm{Hom}(x, y)$. The zero morphisms is denoted by $0 : x \to y$.
4. The **kernel of a morphism** $f : x \to y$ is a morphism $i : z \to x$ such that $fi = 0$ and satisfies the following universal property: Every morphism $i' : z' \to x$, factors through i, such that $i'f = 0$, i.e. there is a unique morphism $j : z' \to z$ such that $ij = i'$.
5. The **cokernel of a morphism** $f : x \to y$ is a morphism $p : y \to t$ such that $pf = 0$ and satisfies the following property: For every morphism $p' : y \to t'$ such that $q'f = 0$ there is a unique $q : t \to t'$ such that $p' = qp$.

Example 138. In the category of R-modules, the kernel of a morphism $f : B \to C$ is the inclusion homomorphism $i : \ker f \hookrightarrow B$ where $\ker f = \{b \in B \,|\, f(b) = 0\}$. Show that the cokernel of f is the quotient homomorphism

$$p : C \to C/\mathrm{im}\, f$$

where $\mathrm{im}\, f = \{f(b) \,|\, b \in B\} \subseteq C$.

Exercise 139 (Weibel [4])**.** Prove the following statements.

1. If $i_1 : A_1 \to B$ and $i_2 : A_2 \to B$ are two kernels of a morphism $f : B \to C$, then $A_1 \cong A_2$ in a category $\mathscr{C}$.
2. The kernel $i : A \to B$ of a morphism $f : B \to C$ is monic.
3. Let $\mathscr{A}$ be an *Ab*-category and $f : B \to C$ a morphism. Show that
   (a) f is monic if and only if for every nonzero $e : A \to B$, $fe \neq 0$.
   (b) f is an epi if and only if for every nonzero $g : C \to D$, $gf \neq 0$.
4. In an abelian category
   (a) $f : B \to C$ is monic if and only if $\ker f$ is zero morphism.
   (b) $f : B \to C$ is epi if and only if $\mathrm{coker}\, f$ is zero morphism.

Now we are ready to give the definition of an abelian category.

Definition 140. An **abelian category** is an additive category $\mathscr{A}$ such that

AB1 Every map in $\mathscr{A}$ has a kernel and cokernel.

AB2 Every monic in $\mathscr{A}$ is the kernel of its cokernel.

AB3 Every epi in $\mathscr{A}$ is the cokernel of its kernel.

Exercise 141. Prove the following statements.

1. R-module homomorphism $f : M \to N$ is monic iff it is injective i.e. $\ker f = 0$.
2. R-module homomorphism $f : M \to N$ is epi iff it is surjective i.e. $\operatorname{coker} f = 0$.

Proposition 142. *For any ring R, the category of R-modules is an abelian category.*

Proof. We have already observed that R-Mod is an additive category with zero object the zero module. Now observe that the condition AB1 is clear. AB2, every monic $f : M \to N$ is an injective map. The quotient map $N \to N/\operatorname{im}(f)$ is the cokernel of f. The kernel of $N \to N/\operatorname{im}(f)$ is equal to $\operatorname{im}(f) \hookrightarrow N$. Since $M \cong \operatorname{im}(f)$ as R-module we can conclude that $f : M \to N$ is the kernel of its cokernel. Note that kernel and cokernel of morphisms are unique up to isomorphism, so called something "the kernel" is a little bit misleading. We can show AB3 in a similar way. □

Another important example of abelian categories is the category of chain complexes over an abelian category. We first consider the chain complexes over a ring.

Let R be a ring with unity. Consider the category of all chain complexes over R with morphisms given by the chain maps. The category of chain complexes is an *Ab*-category where the addition of two chain maps

$$g_* + g'_* : B_* \to C_*$$

is defined by $(g_* + g'_*)_n = g_n + g'_n$ for all $n \in \mathbb{Z}$. In category of chain complexes, there is a zero object which is the zero complex (i.e. the complex with $C_n = 0$ for all n). We have the following: The category of chain complexes over R, denoted by $Ch(R)$, is an additive category where the product is defined by $(C_* \oplus D_*)_n = C_n \oplus D_n$ with boundary maps $d_n(x, y) = d_n^C(x) + d_n^D(y)$ for every $x \in C_n$ and $y \in D_n$. In fact the category $Ch(R)$ is an abelian category where the kernel of a chain map $f_* : A_* \to B_*$ is defined by the incluson of the subcomplex $\ker f_* \to A_*$ and the cokernel is defined as the quotient map $B_* \to B_*/\operatorname{im} f_*$.

Exercise 143. Write the details of the proof that $Ch(R)$ is an abelian category.

We will now extend the definition of chain complexes over R-modules to chain complexes over any abelian category. Given an abelian category $\mathscr{A}$, we can define chain complexes and chain maps in $\mathscr{A}$. Any chain complex in $\mathscr{A}$ is a sequence of morphisms.

$$C_* : \cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \ldots$$

such that $C_n \in \mathrm{Ob}(\mathscr{A})$ and $d_n \in \mathrm{Hom}_{\mathscr{A}}(C_n, C_{n-1})$ for all n satisfying $d_{n-1}d_n = 0$ in $\mathrm{Hom}_{\mathscr{A}}(C_n, C_{n-2})$. We have the following:

Theorem 144. *The category $Ch(\mathscr{A})$ of chain complexes in an abelian category $\mathscr{A}$ is an abelian category.*

Proof. See Theorem 1.2.3 in [4]. □

A.2 Adjoint Pairs

Definition 145. Let $L : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ be two additive functors. We say (L, R) is an adjoint pair if there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(A, R(B)).$$

When (L, R) is an adjoint pair we say L is left adjoint to R and R is right adjoint to L.

The most important adjoint pair for additive functors is the tensor and Hom-functor pair. Let R and S be two rings, and B be an R-S-bimodule. Then for every right R-module A, the tensor product $A \otimes_R B$ is a right S-module with the S-action given by $(a \otimes b)s = (a \otimes bs)$. This defines an additive functor $- \otimes_R B$ from right R-modules to right S-modules. Similarly, for every right S-module C, the Hom-set $\mathrm{Hom}_S(B, C)$ has a right R-module structure given by the action $(fr)(b) = f(rb)$ for every $r \in R$, $b \in B$. This defines an additive functor $\mathrm{Hom}_S(B, -)$ from right S-modules to right R-modules.

Proposition 146. *Let R and S be two rings and B be an R-S-bimodule. Then the functor $\mathrm{Hom}_S(B, -)$ is right adjoint to the functor $- \otimes_R B$. That is for every right R-module A and right S-module C there is natural isomorphism*

$$\psi : \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C).$$

Proof. For every S-homomorphism $f : A \otimes_R B \to C$, let $\Psi(f)$ be equal to the homomorphism $g : A \to \mathrm{Hom}_S(B, C)$ defined by $g(a)(b) = f(a \otimes b)$. For every $s \in S$, we have

$$g(a)(bs) = f(a \otimes bs) = f(a \otimes b)s) = f(a \otimes b)s = g(a)(b)s.$$

So, $g(a)$ is an S-homomorphism. For every $r \in R$, we have

$$g(ar)(b) = f(ar \otimes b) = f(a \otimes rb) = g(a)(rb) = (g(a)r)(b)$$

This shows that g is a right R-module homomorphism.

The inverse of ψ^{-1} is defined as the homomorphism that takes $g : A \to \mathrm{Hom}_S(B, C)$ to $f : A \otimes_R B \to C$ where for every $a \in A$ and $b \in B$, $f(a \otimes b) = g(a)(b)$. It is easy to check that f is well defined and it is an S-module homomorphism. By definition ψ and ψ^{-1} are inverse to each other. Naturality is also easy to show. □

For adjoint pairs we have the following property:

Theorem 147. *Let $L : \mathscr{A} \to \mathscr{B}$ be an additive functor which is left adjoint to the functor $R : \mathscr{B} \to \mathscr{A}$. Then L is right exact and R is left exact.*

Proof. See Theorem 2.6.1 in [4] □

As a corollary we obtain the following:

Corollary 148. *For every R-S-bimodule B, the tensor product functor*

$$- \otimes_R B : R\text{-}mod \to S\text{-}mod$$

is right exact.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic press, 1972.

[2] K. Brown, *Cohomology of Groups*, Springer-Verlag GTM 87, 1982.

[3] J. F. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, Graduate Studies in Mathematics, Vol. 35, Amer. math. Soc. 2000.

[4] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.