1. $\partial/\partial p_k \; \{-k_B \sum p_i \ln p_i + \Lambda_1(U - \sum_i E_i p_i) + \Lambda_2(1 - \sum_i p_i) + \Lambda_3(M - \sum_i p_i m_i)\} = 0$

$-k_B(\ln p_k + 1) - \Lambda_1 E_k - \Lambda_2 - \Lambda_3 m_k = 0$

$p_k = \exp(-\beta E_k - \beta h m_k)/Z \qquad Z = \sum_k \exp(-\beta E_k - \beta h m_k)$

For the given system, (atoms are not interacting) $Z = (Z_1)^N$.
There are 4 states for each atom: $(E_1, \mu)$, $(E_2, \mu)$, $(E_1, -\mu)$, and $(E_2, -\mu)$.
Then

$$
\begin{aligned}
Z_1 &= \exp[-\beta(E_1 + h\mu)] + \exp[-\beta(E_1 - h\mu)] + \exp[-\beta(E_2 + h\mu)] + \exp[-\beta(E_2 - h\mu)] \\
&= [\exp(-\beta E_1) + \exp(-\beta E_2)] \cdot [\exp(-\beta h\mu) + \exp(+\beta h\mu)] \\
&= 2\cosh(\beta h\mu)[\exp(-\beta E_1) + \exp(-\beta E_2)]
\end{aligned}
$$

$M = < m > = (1/\beta)\; \partial/\partial h \; \ln Z$

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2. (a) Single particle energies: $\epsilon_1 = (\hbar\pi/L)^2/2m$ and $\epsilon_2 = 4\epsilon_1$, $\quad \epsilon_3 = 9\epsilon_1$
Occupation of single-particle states with degeneracies:

| $n=1$ | $n=2$ | $n=3$ | $E/\epsilon_1$ | degeneracy |
|---|---|---|---|---|
| ↑↓ | ↑↓ | | 10 | 1 |
| ↑↓ | ↑ | ↑ | 15 | |
| ↑↓ | ↓ | ↑ | 15 | |
| ↑↓ | ↑ | ↓ | 15 | |
| ↑↓ | ↓ | ↓ | 15 | 4 |
| ↑ | ↑↓ | ↑ | 18 | |
| ↑ | ↑↓ | ↓ | 18 | |
| ↓ | ↑↓ | ↑ | 18 | |
| ↓ | ↑↓ | ↓ | 18 | 4 |
| ↑↓ | | ↑↓ | 20 | 1 |

(b) $Z = \exp(-10\beta\epsilon_1) + 4\exp(-15\beta\epsilon_1) + 4\exp(-18\beta\epsilon_1) + \exp(-20\beta\epsilon_1)$

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3. We use the geometry of the circle instead of the sphere: Number of states with wavenumber between $k$ and $k + dk$: $2 \times (2\pi k dk/4)/(\pi/L)^2 = Akdk/\pi$ where $A$ is the area of the region and the factor of 2 is for spin degeneracy. At zero temperature, all states up to the Fermi wavevector $k_F$ will be filled: $N = \int_o^{k_F} Akdk/\pi = Ak_F^2/2\pi$. Then $k_F^2 = 2\pi N/A$ and $E_F = \hbar^2 k_F^2/2m = (\hbar^2/2m)(2\pi N/A)$.