First Exam Solutions — Bilkent University

Spring 2023 — Phys 334

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1. $Z = \int \cdots \int \exp(-\beta H\{p,q\}) d^{3N}q\, d^{3N}p/(N! h^{3N})$ with $H\{p,q\} = \sum_{i=1}^{3N} p_i^2/2m - \sum_{i=1}^{N} mgz_i$

So, $Z = (Z_1)^N/(N! h^N)$ with $Z_1 = [\int \exp(-\beta p^2/2m)dp]^3 \int_0^h \exp(-\beta mgz)dx \int dy \int dz$

$= [2m\pi/\beta]^{3/2} A(1 - \exp(-\beta mgh))/(\beta mg)$

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2. $P = U/3V = (bVT^4)/3V = bT^4/3$ which is constant for constant T.

Work done by system: $W = \int_{V_o}^{2V_o} PdV = bV_oT_o^4/3$

Change in the internal energy of the system: $\Delta U = (b\, 2V_o\, T_o^4) - (bV_oT_o^4) = bV_oT_o^4$

Heat must supply these energies: $Q = \Delta U + W = 4bV_oT_o^4/3$

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3. $$\frac{\partial}{\partial t}\mathrm{P} = \omega \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \mathrm{P}$$

with the characteristic equation for the matrix:

$2 - (\lambda+1)(\lambda+2)(\lambda+1) = 0$

$\lambda(\lambda^2 + 4\lambda + 5) = 0$

which results in $\lambda = 0$ and $\lambda = -2 \pm i$

Complex eigenvalues mean that the system will have a decaying exponential approach to equilibrium.

$\lambda = 0$ corresponds to the equilibrium eigenvalue and results in the corresponding right eigenvector

$\frac{1}{5}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ So, $p_1(t)$ will start from $p_1(0) = 1$ and decay to $p_1(\infty) = 2/5$ with a form $\exp(-2\omega t) \begin{Bmatrix} \sin(\omega t) \\ \cos(\omega t) \end{Bmatrix}$

4.

(a) $Z = (Z_1)^N \qquad Z_1 = \sum_n exp(-\beta E_n) = \sum_{n=0}^{m-1} exp(-\beta n E_o) = [1 - \exp(-\beta m E_o)]/[1 - \exp(-\beta E_o)]$

(b) $U = [E] = -\partial/\partial\beta\ \ln Z = E_o \exp(-\beta E_o)/[1 - \exp(-\beta E_o)] - mE_o \exp(-\beta m E_o)/[1 - \exp(-\beta m E_o)]$

$= E_o/[\exp(\beta E_o) - 1]\ -\ mE_o/[\exp(\beta m E_o) - 1]$

(c) For high temperatures, β is small, expanding the exponentials,

$U \approx E_o/[1 + (\beta E_o) + (\beta E_o)^2/2 + \cdots - 1]\ -\ mE_o/[1 + (\beta m E_o) + (\beta m E_o)^2/2 + \cdots - 1]$

$\approx (1/[1 + \beta E_o/2]\ -\ 1/[1 + \beta m E_o/2])/\beta$

$\approx E_o(m-1)/2$. At high temperatures all states are equally occupied and the average energy becomes the average energy of all of the states.