1. (a) $Z = \int d^{3N_1}q_1\ d^{3N_2}q_2\ d^{3N_1}p_1\ d^{3N_2}p_2\ \exp(-\beta\sum p_{1i}^2/2m_1 - \beta\sum p_{2i}^2/2m_2)/(N_1!N_2!h^{3(N_1+N_2)})$

Each Gaussian integral gives a factor $\int dp\,\exp(-\beta p^2/2m) = \sqrt{2m\pi/\beta}$ and each triple position integral gives a factor of V. We then have

$Z = (2m_1\pi/\beta)^{3N_1/2}\ (2m_2\pi/\beta)^{3N_2/2}\ V^{N_1+N_2}\ /\ (N_1!N_2!h^{3(N_1+N_2)})$

(b) Number of quadratic degrees of freedom $= 3(N_1+N_2) \Longrightarrow U = 3(N_1+N_2)k_BT/2$

(c) $A = -\ln Z/\beta$ and $P = -\frac{\partial A}{\partial V})_T$

$\Longrightarrow P = \frac{\partial}{\partial V}(\ln Z)/\beta = \frac{\partial}{\partial V}(\ln V^{N_1+N_2} + \ln\cdots)/\beta = (N_1+N_2)/(V\beta)$

or $PV = (N_1+N_2)k_BT$.

2.(a) Probability that n particles have energy E_1 and $N-n$ particles have energy E_2 is

$P(n) = \exp(-\beta nE_1 - \beta(N-n)E_2)/Z$ with $Z = \sum_{k=0}^{N}\exp(-\beta kE_1 - \beta(N-k)E_2)$

or, in our case, as $E_1 = 0$ and $E_2 = \epsilon$, we have $P(n) = \exp[-\beta(N-n)\epsilon]/Z$ with

$Z = \sum_{k=0}^{N}\exp(-\beta(N-k)\epsilon) = \exp(-\beta N\epsilon)\{1-\exp[\beta(N+1)\epsilon]\}/[1-\exp(\beta\epsilon)]$

$= [1-\exp(-\beta(N+1)\epsilon]/[1-\exp(-\beta\epsilon)]$ so that

$P(n) = \exp[-\beta(N-n)\epsilon][1-\exp(-\beta\epsilon)]/[1-\exp(-\beta(N+1)\epsilon]$

(b) $P(N) = [1-\exp(-\beta\epsilon)]/[1-\exp(-\beta(N+1)\epsilon] \longrightarrow 1-\exp(-\beta\epsilon)$ as $N\to\infty$.

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3. We have

$$\begin{aligned} Z &= \sum_{\{S_i=\pm1\}} \exp(KS_1S_2)\exp(K'S_2S_3)\exp(KS_3S_4)\cdots\exp(K'S_NS_1) \\ &= \sum_{\{S_i=\pm1\}} T_{S_1,S_2}\quad U_{S_2,S_3}\quad T_{S_3,S_4}\quad \cdots\quad U_{S_N,S_1} \\ &= \text{Tr }(TU)^{N/2} \text{ where} \end{aligned}$$

$$T = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \text{ and } U = \begin{pmatrix} e^{K'} & e^{-K'} \\ e^{-K'} & e^{K'} \end{pmatrix} \text{ so that } TU = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$\text{with}\quad \begin{array}{l} a = e^{K+K'} + e^{-K-K'} = 2\cosh(K+K') \\ b = e^{K-K'} + e^{K'-K} = 2\cosh(K-K') \end{array}$$

Eigenvalues of TU may be determined from $(a-\lambda)^2 - b^2 = 0$ which yields $\lambda_\pm = a \pm b$.
We then get $Z = \lambda_+^{N/2} + \lambda_-^{N/2}$.