
1.(a) The density matrix is $\rho = |1\ 1\rangle\, 0.5\, \langle 1\ 1| + |\uparrow\downarrow\rangle\, 0.2\, \langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\, 0.2\, \langle\downarrow\uparrow| + |1\ -1\rangle\, 0.1\, \langle 1\ -1|$

To find the matrix elements of ρ and S_x in the total angular momentum basis, we find the elements $\langle u_i|\, \rho\, |u_j\rangle$, and $\langle u_i|\, S_x\, |u_j\rangle = \langle u_i|\, (S_+ + S_-)\, |u_j\rangle$.

Using the order $|u_o\rangle = |0\ 0\rangle$, $|u_1\rangle = |1\ 1\rangle$, $|u_2\rangle = |1\ 0\rangle$, $|u_3\rangle = |1\ -1\rangle$, we get

$$\rho = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix} \qquad S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

Note that $\mathrm{Tr}(\rho) = 1$ and $[S_x] = \mathrm{Tr}(\rho S_x)$ so that

$$[S_x] = \frac{\hbar}{\sqrt{2}}\, \mathrm{Tr} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0.2 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0 \end{pmatrix} = 0$$

---

2. Remembering that $J^2 = (\vec{L} + \vec{S})^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$ so that $\vec{L}\cdot\vec{S} = (J^2 - L^2 - S^2)/2$.

This means $\vec{L}\cdot\vec{S}$, commutes with, and therefore shares the same eigenvectors with J^2, L^2, and S^2. We will have, for $l = 1$, total angular momentum $j = 3/2$ or $j = 1/2$. The possible states and their energies are then

$j = 1/2$ 2-fold degenerate	$\left\{ \left\|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle,\ \left\|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle \right.$	$j = 3/2$ 4-fold degenerate	$\left\{ \left\|\tfrac{3}{2}\ \tfrac{3}{2}\right\rangle,\ \left\|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle,\ \left\|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle,\ \left\|\tfrac{3}{2}\ -\tfrac{3}{2}\right\rangle \right.$

with energies $E_{1/2} = \alpha\hbar^2[\frac{1}{2}(1+\frac{1}{2}) - 1(1+1) - \frac{1}{2}(1+\frac{1}{2})]/2 = -\alpha\hbar^2$

and $E_{3/2} = \alpha\hbar^2[\frac{3}{2}(1+\frac{3}{2}) - 1(1+1) - \frac{1}{2}(1+\frac{1}{2})]/2 = \alpha\hbar^2[\frac{15}{4} - 2 - \frac{3}{4}]/2 = \alpha\hbar^2/2$.

---

3. Let's choose $\psi_{var} = A\exp(-a|x|)$. Normalization will give $A^2/2a = 1$.

$\langle d^2/dx^2\rangle = \int_{-\infty}^{\infty} \psi^* d^2\psi/dx^2\ dx = -\int_{-\infty}^{\infty} d\psi^*/dx\ d\psi/dx\ dx = -2\int_o^\infty |d\psi/dx|^2\ dx = -A^2 \int_o^\infty a^2 \exp(-2ax)$

$\qquad = -2a \cdot a^2 \cdot (1/2a) = -a^2$ so that $\left\langle -(\hbar^2/2m) d^2/dx^2 \right\rangle = a^2\hbar^2/2m$

$\langle V\rangle = \int_{-\infty}^{\infty} \psi^* k|x|^3 \psi\ dx = 2A^2 \int_o^\infty kx^3 \exp(-2ax)\ dx = 2(2a)k(3!)/(2a)^4 = 3k/(4a^3)$.

Finding the extremum $\frac{\partial}{\partial a}\left[a^2\hbar^2/2m + 3k/(4a^3)\right] = a\hbar^2/m - 9k/(4a^4) = 0 \quad \Longrightarrow \quad a = \left(9mk/4\hbar^2\right)^{1/5}$

Substituting, $E_{est} = \left(9mk/4\hbar^2\right)^{2/5}\hbar^2/2m + \left(9mk/4\hbar^2\right)^{-3/5} 3k/4 = \hbar^{-6/5} k^{2/5} m^{-3/5}\left[(9/4)^{2/5}/2 + 3(4/9)^{3/5}/4\right]$