Final Exam Solutions
Bilkent University

Spring 2026
Phys 326

1.(a) We have $|2\ 1\rangle\ \left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \sqrt{\frac{4}{5}}\left|\frac{5}{2}\ \frac{3}{2}\right\rangle - \sqrt{\frac{1}{5}}\left|\frac{3}{2}\ \frac{3}{2}\right\rangle$ and $|2\ 1\rangle\ \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|\frac{5}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$ and since $|\downarrow_x\rangle = (|\uparrow_z\rangle - |\downarrow_z\rangle)/\sqrt{2}$, we have $|2\ 1\rangle\ |\downarrow_x\rangle = \sqrt{\frac{4}{10}}\left|\frac{5}{2}\ \frac{3}{2}\right\rangle - \sqrt{\frac{1}{10}}\left|\frac{3}{2}\ \frac{3}{2}\right\rangle + \sqrt{\frac{2}{10}}\left|\frac{5}{2}\ \frac{1}{2}\right\rangle + \sqrt{\frac{3}{10}}\left|\frac{3}{2}\ \frac{1}{2}\right\rangle$.
So that we have probability $(4+2)/10 = 0.6$ for $j = 5/2$ (measurement $\hbar^2 5/2(1+5/2) = \hbar^2 35/4$)
and probability $(1+3)/10 = 0.4$ for $j = 3/2$ (measurement $\hbar^2 3/2(1+3/2) = \hbar^2 15/4$).

(b) Since $L_z = \hbar$ and we have equal probabilities for $S_z = \pm\hbar/2$, we have probability 0.5 for $J_z = 3\hbar/2$ and 0.5 for $J_z = \hbar/2$.

(c) Since $J_z = 3\hbar/2$, we must have $m = 1$. We are given that the energy state is $n = 2$, so $l = 1$ is the only state that can have $m = 1$. The energy state of the electron must be $|2\ 1\ 1\rangle$.

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2. Initially, the electron is in the ground state of the Helium potential, with the wavefunction given by $\psi(\vec{r}, t=0) = 2(2/a_o)^{3/2}\exp(-2r/a_o)\,|0\ 0\rangle$. At time $t = 0^+$, the potential is that of a single proton, and the eigenfunctions are those of a Hydrogen atom. We can expand $\psi(\vec{r}, t=0)$ in terms of the new eigenfunctions: $\psi(\vec{r}, t=0) = \sum_{nlm} c_{nlm} R_{nl}(r)\,|l\ m\rangle$. The coefficient $|c_{100}|^2$ will give us the probability that the electron is in the ground state. We can find $c_{100} = \int_0^\infty dr\ r^2 \psi(\vec{r}, t=0) R_{10}(r)$

$c_{100} = 2(2/a_o)^{3/2} 2(1/a_o)^{3/2} \int_0^\infty dr\ r^2 \exp(-2r/a_o)\exp(-r/a_o) = 4\cdot 2^{3/2}\int_0^\infty dr/a_o\ (r/a_o)^2\exp(-3r/a_o)$

$c_{100} = 4\cdot 2^{3/2}(2/27) = 16\sqrt{2}/27$ so that $p = |c_{100}|^2 = 512/729 \approx 0.7$

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3. (a) $E_{00}^{(1)} = \alpha\int u_o^*(x_1)u_o^*(x_2)[x_1^2 + x_2^2 - 2x_1x_2]u_o(x_1)u_o(x_2)dx_1dx_2 = 2\alpha < x^2 >_{oo} = \alpha x_o^2/2$

(b) For total energy to be $3\hbar\omega_c$ we can either have both particles in $n = 1$ or one particle in $n = 0$ and the other in $n = 2$. So, that energy is two-fold degenerate.

(c) The degenerate wavefunctions are $\psi_\alpha = u_1(x_1)u_1(x_2)$ and $\psi_\beta = [u_o(x_1)u_2(x_2) + u_2(x_1)u_0(x_2)]/\sqrt{2}$.
We need to find the matrix elements

$V_{\alpha\alpha} = \alpha\int u_1^*(x_1)u_1^*(x_2)[x_1^2 + x_2^2 - 2x_1x_2]u_1(x_1)u_1(x_2)dx_1dx_2 = 2\alpha < x^2 >_{11} = \alpha 3x_o^2/2$

$V_{\beta\beta} = (\alpha/2)\int[u_o^*(x_1)u_2^*(x_2) + u_2^*(x_1)u_0^*(x_2)][x_1^2 + x_2^2 - 2x_1x_2][u_o(x_1)u_2(x_2) + u_2(x_1)u_0(x_2)]dx_1dx_2$

Note that $< x >_{nn}$ and $< x >_{02}$ averages are zero. Also, $< x^2 >_{nn} = (1+2n)x_o^2/4$.
We then have $V_{\beta\beta} = (\alpha/2)(< x^2 >_{oo} + < x^2 >_{22})\cdot 2 = \alpha 3x_o^2/2$.

$V_{\alpha\beta} = (\alpha/\sqrt{2})\int u_1^*(x_1)u_1^*(x_2)[x_1^2 + x_2^2 - 2x_1x_2][u_o(x_1)u_2(x_2) + u_2(x_1)u_0(x_2)]dx_1dx_2$

Now note that averages $< x^2 >_{10}$ and $< x^2 >_{12}$ are zero but $< x >_{10} = x_o/2$ and $< x >_{12} = \sqrt{2}x_o$ so that $V_{\alpha\beta} = (\alpha/\sqrt{2})[-2(\sqrt{2}x_o)(x_o/2)] = -\alpha x_o^2$

The corresponding matrix is $\alpha x_o^2\begin{pmatrix} 3/2 & -1 \\ -1 & 3/2 \end{pmatrix}$ with eigenvalues $\alpha x_o^2(3/2 \pm 1)$