Second Exam Solutions | Bilkent University

Fall 2023 | Phys 325

1.

(a) $u_1(x) = a_+ u_o(x) = (x/x_o - iP/p_o)u_o(x) = [x/x_o - (\hbar/p_o)\partial/\partial x]\ A\exp(-x^2/x_o^2)$
$= [x/x_o - (\hbar/p_o)(-2x/x_o^2)]\ A\exp(-x^2/x_o^2) = (2x/x_o)\ u_o(x)$

(b) $\psi(x,t) = [u_o(x)\exp(-itE_o/\hbar) + u_1(x)\exp(-itE_1/\hbar)]$
$j(0) = (\hbar/m)\ \text{Im}\ [\psi^* \partial\psi/\partial x]_o$
We have $u_o(0) = A \quad u_1(0) = 0 \quad u_o'(0) = 0$ and $u_1'(0) = u_o(0)/x_o = A/x_o$
so that the only finite term is $j(0) = (\hbar/m)\ \text{Im}\ [u_o^*(0)\exp(itE_o/\hbar)u_1'(0)\exp(-itE_1/\hbar)]$
which yields $j(0) = -|A|^2\hbar/(x_o m)\sin(\omega_c t)$

(c) $g(k) = |A|^2 \int 2xexp(-x^2/x_o^2)\exp(-ikx)dx/(x_o\sqrt{2\pi})$ Let us omplete the square of the
exponential: $\exp(-x^2/x_o^2)\exp(-ikx) = \exp[-(x - ikx_o^2/2)^2/(x_o)^2 - k^2x_o^2/2]$
making a variable transformation $u = x - ikx_o^2/2$ we get
$g(k) = 2|A|^2\exp(-k^2x_o^2/2)\int(u + ikx_o^2)\exp(-u^2/x_o^2)du/(x_o\sqrt{2\pi})$
First integral is zero due to symmetry, and the second term gives $g(k) = 2|A|^2\exp(-k^2x_o^2/2)ik$

2.

(a) a is unitless.

(b) Eigenvalues are $\lambda_1 = (1+a)\epsilon$ and $\lambda_2 = (1-a)\epsilon$.

(c) Eigenvectors are $\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

(d) Note that $\psi(0) = \begin{pmatrix}1\\0\end{pmatrix} = (\psi_1 + \psi_2)/\sqrt{2}$

so that $\psi(t) = [\psi_1\exp(-it\lambda_1/\hbar) + \psi_2\exp(-it\lambda_2/\hbar)]/\sqrt{2}$

3.

(a) $A^+ = (XP + PX)^+ = P^+X^+ + X^+P^+ = PX + XP = A$

(b) $[A, X] = [XP + PX, X] = X[P, X] + [P, X]X = -2i\hbar X$

(c) $[A, P] = [XP + PX, P] = [X, P]P + P[X, P] = 2i\hbar P$

(d) $A = XP + PX = (x_o p_o/4i)[(a_+ + a_-)(a_- - a_+) + (a_- - a_+)(a_+ + a_-)]$
$= (x_o p_o/4i)2[a_-^2 - a_+^2]$ which leads to $< n|A|n >= 0$.