
1.(a) $\int (A^* u_1^* + u_2^*/2)(A^* u_1 + u_2/2)dx = |A|^2 + 1/4 = 1 \quad \Longrightarrow \quad A = \sqrt{3}/2$

(b) $< p > (t) = \int \psi(x,t)^* (-i\hbar\, \partial/\partial x)\psi(x,t)dx$

note that:

$\int u_1^* (\partial/\partial x) u_1 dx \sim \int_{-a/2}^{a/2} \cos(\pi x/a)\sin(\pi x/a)dx = 0$

$\int u_2^* (\partial/\partial x) u_2 dx \sim \int_{-a/2}^{a/2} \sin(2\pi x/a)\cos(2\pi x/a)dx = 0$

$\int u_1^* (\partial/\partial x) u_2 dx = -\int u_2^* (\partial/\partial x) u_1 dx = (2/a)(2\pi/a) \int_{-a/2}^{a/2} \cos(\pi x/a)\cos(2\pi x/a)dx$

$\quad = (2/a)(\pi/a) \int_{-a/2}^{a/2} [\cos(\pi x/a) + \cos(3\pi x/a)]\, dx = (2/a)(\pi/a)\, [2/(\pi/a) - 2/(3\pi/a)] = 8/(3a)$

$< p > (t) = (\sqrt{3}/2)\ (1/2)\ [8/(3a)]\ (-i\hbar)\ (\exp[i(E_1 - E_2)t/\hbar] - \exp[i(E_2 - E_1)t/\hbar])$

$\quad = -4\hbar/(\sqrt{3}a)\ \sin[(E_2 - E_1)t/\hbar]$ with $E_n = \hbar^2\pi^2 n^2/(2ma^2)$.

2.(a) $\int \psi^*(x,0)\psi(x,0)dx = 1 \quad \Longrightarrow \quad |A|^2 \int_o^a \sin^2(\pi x/a)dx = |A|^2 a/2 = 1 \quad \Longrightarrow \quad A = \sqrt{2/a}$

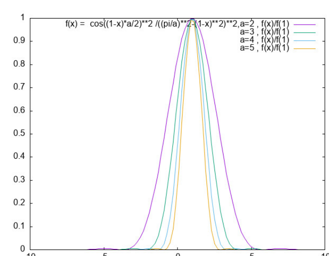
(b) $< p >= -i\hbar \int \psi^*\ (\partial/\partial x)\ \psi dx$

$\quad = -2i\hbar/a \int_o^a \sin(\pi x/a)\exp(-ik_o x)[(\pi/a)\cos(\pi x/a) + ik_o \sin(\pi x/a)]\exp(-ik_o x)dx$

First integral is zero due to symmetry, the second one gives $< p >= \hbar k_o$ as expected.

(c)
$$\begin{aligned} g(k) &= (1/\sqrt{a\pi}) \int_o^a [\exp(i\pi x/a) - \exp(-i\pi x/a)]/(2i)\ \exp(ik_o x - ikx)dx \\ &= \frac{1}{\sqrt{a\pi}}\frac{1}{2i}\left(\frac{\exp[i\pi + i(k_o - k)a] - 1}{i(\pi/a + k_o - k)} - \frac{\exp[-i\pi + i(k_o - k)a] - 1}{i(-\pi/a + k_o - k)}\right) \\ &= \frac{1 + \exp[i(k_o - k)a]}{2\sqrt{a\pi}}\frac{-2\pi/a}{(\pi/a)^2 + (k_o - k)^2} \end{aligned}$$

(d) $\qquad |g(k)|^2 \sim \cos^2[(k_o - k)a/2]/[(\pi/a)^2 + (k_o - k)^2]^2$



The function peaks at $k = k_o$, the width of the peak is related to a. Bigger the a, narrower the peak. At right is a plot of $g(k)/g(1)$ for $k_o = 1$ and $a = 2, 3, 4, 5$.

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3. (a) We will be determining bound states.

(b) We will have

$\quad u(x) = A\cos(qx) + B\sin(qx)$ for $0 < x < a/2$ with $q = \sqrt{2m(E - V_o}/\hbar$

$\quad u(x) = C\cos(kx) + D\sin(kx)$ for $a/2 < x < a$ with $k = \sqrt{2mE}/\hbar$

$\quad u(x) = 0$ elsewhere.

(c) Boundary conditions at $x = 0$: $u(0) = 0 \quad \Longrightarrow \quad u(x) = B\sin(qx)$ for $0 < x < a/2$

Boundary conditions at $x = a$: $u(a) = 0 \quad \Longrightarrow \quad u(x) = C\sin[k(x - a)]$ for $a/2 < x < a$

Boundary conditions at $x = a/2$: u is continious $\Longrightarrow B\sin(qa/2) = -C\sin(ka/2)$

u' is continious $\Longrightarrow Bq\cos(qa/2) = Ck\cos(ka/2)$

(d) Dividing the two equations into one another, we get

$$\tan(qa/2)/q = -\tan(ka/2)/k$$

or

$$\tan\left(\frac{a}{2\hbar}\sqrt{2m(E - V_o)}\right) = -\sqrt{\frac{E - V_o}{E}}\ \tan\left(\frac{a}{2\hbar}\sqrt{2mE}\right)$$

Solutions to this transcendental equation will give the energy eigenvalues.