

1.(a) $\alpha \cdot \text{distance} = \rho \implies$ units of α is Coulombs/m⁴.

(b) We will consider cylindrical closed surfaces S coaxial with the z -axis and with radius r and length L . The flux of the E-field through the flat circular faces of S will be zero as the field is parallel to those surfaces. Flux of the field through the curved part of S will be $2\pi rLE = Q_{\text{enc}}/\epsilon_o$. We also have

$$Q_{\text{enc}} = \begin{cases} \int_0^r \alpha(r - 2a/3)2\pi rL dr = 2\pi L\alpha(r^3 - ar^2)/3 & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

so that the electric field (in the radially outward direction is)

$$E = \begin{cases} \alpha(r^2 - ar)/(3\epsilon_o) & \text{for } r < a \\ 0 & \text{for } r > a \end{cases} .$$

(c) The potential is then

$$V = \begin{cases} 0 & \text{for } r > a \\ \int_r^a \alpha(r^2 - ar)/(3\epsilon_o) dr = \alpha[(a^3 - r^3)/3 - a(a^2 - r^2)/2]/(3\epsilon_o) & \text{for } r < a \end{cases} .$$

2.(a) For a zero-potential sphere, for the outside potential, one needs to consider Q at $x = a$ and its image $-QR/a$ at $x = R^2/a$. But this spherical surface will contain a charge $-QR/a$ on it. For the “given” amount of charge $-Q$ on the sphere, we need to place a charge $+QR/a - Q$ charge at the center of the sphere. This extra charge generates a potential $V_{\text{sphere}} = Q(R/a - 1)/(4\pi\epsilon_o R)$.

(b) Potential on the x -axis due to these charges will be

$$V(x) = \frac{Q}{4\pi\epsilon_o} \left(\frac{R/a - 1}{x} - \frac{R/a}{|x - R^2/a|} + \frac{1}{|x - a|} \right).$$

(c) For $x \gg a$

$$\begin{aligned} V(x \rightarrow \infty) &= \frac{Q}{4\pi\epsilon_o x} \left(R/a - 1 - \frac{R/a}{1 - R^2/ax} + \frac{1}{1 - a/x} \right) \\ &\sim \frac{Q}{4\pi\epsilon_o x} \left(R/a - 1 - (R/a)(1 + R^2/ax) + 1 + a/x \right) = \frac{Q(a - R^3/a^2)}{4\pi\epsilon_o x^2} \end{aligned}$$

(d) Since we have two objects with $+Q$ and $-Q$ charge, the potential at large distances behaves like that of a dipole with $p = Qa(1 - R^3/a^3)$. Note that as the sphere radius R decreases, $p \rightarrow Qa$.

3.
$$V(x, y) = V_o \frac{\sinh(\pi x/b)}{\sinh(\pi a/b)} \sin(\pi y/b)$$

