First Exam Solutions — Bilkent University
Spring 2022 — Phys 315

1.(a) Total charge = 0 $\implies$ $\rho\,[4\pi R^3/3] = -\sigma\,[4\pi R^2]$ $\implies$ $\sigma = -\rho R/3$

(b) For a spherically symmetric potential, you can use the Gauss's Law, which will give you the result that the electric field at any radius will be equivalent to the electric field E produced by a point charge (with magnitude equal to the amount of charge contained within that radius) at the center of the sphere. Since we are given that the total charge is zero, we have

$E = 0$ for $r > R$.

For $r < R$ we have $E = (\rho 4\pi r^3/3)/(4\pi\ \epsilon_o r^2) \ = \ \rho r/(3\epsilon_o)$.

For the electric potential, we integrate this field with respect to r to $r = \infty$ (where V=0):

$V(r) = \int_r^\infty 0 \cdot dr + V(\infty) = 0$ for $R < r < \infty$.

$V(r) = \int_r^R \rho r/(3\epsilon_o) dr + V(R) \ = \ \rho(R^2 - r^2)/(6\epsilon_o)$ for $0 < r < R$.

2.(a) Charge on inner surface of cavity is $-q$.

(b) Since the total sphere is uncharged, charge on the outer surface is $+q$.

(d) Inside the cavity, force on q is equivalent to the force on its image:

$$F_q = \frac{q(rq/a)}{4\pi\epsilon_o(r^2/a - a)^2} \text{ (towards right).}$$

From the outside, the sphere looks like a equipotential with a total charge $+q$ on its surface. We can find the force on the external charge Q as a superposition of the forces between Q and its image

$$F_{Q1} = \frac{Q(-RQ/b)}{4\pi\epsilon_o(b - R^2/b)^2} \text{ (towards left)}$$

and the force between Q and a central charge $q + RQ/b$ to achieve total charge q:

$$F_{Q2} = \frac{Q(q + RQ/b)}{4\pi\epsilon_o b^2} \text{ (towards right).}$$

(e) The total force on the external charge Q is $F_Q = F_{Q1} + F_{Q2}$.

(c) Due to Newton's third law, force on the conductor is opposite to those on the charges: $F_C = -F_Q - F_q$.

3.(a) Potential due to a uniformly charged ring or radius r at a distance d on its axis is $Q/(4\pi\epsilon_o\sqrt{r^2 + d^2})$. Potential due to a uniformly charged disc of radius R at a distance d on its axis may be found by integrating over rings:

$$V(d) = \int_0^R \frac{\sigma 2\pi r dr}{4\pi\epsilon_o\sqrt{r^2 + d^2}} = \frac{\sigma}{4\epsilon_o}\int_{\cdots}^{\cdots} \frac{du}{\sqrt{u}} = \frac{\sigma}{4\epsilon_o}\left.\frac{\sqrt{u}}{1/2}\right|_{\cdots}^{\cdots} = \frac{\sigma}{2\epsilon_o}\left.\sqrt{r^2 + d^2}\right|_{r=0}^{r=R} = \frac{\sigma}{2\epsilon_o}\left[\sqrt{R^2 + d^2} - d\right].$$

So, the potential at $+a$ due to the disk $+a$ is $\frac{\sigma R}{2\epsilon_o}$

the potential at $+a$ due to the disk at $-a$ is $-\frac{\sigma}{2\epsilon_o}\left[\sqrt{R^2 + 4a^2} - 2a\right]$

(a) Potential difference will be twice the sum of these potentials: $\Delta V = \frac{\sigma}{\epsilon_o}\left[R + 2a - \sqrt{R^2 + 4a^2}\right]$.

(b) In the limit a/R becomes very small, $\sqrt{R^2 + 4a^2} \sim R$ so that $\Delta V \sim 2a\sigma/\epsilon_o$.

(Note that in this limit the problem reduces to an infinite parallel plate capacitor.)