Some notes on the connection formulas for the WKB approximation:

The "linear potential" part of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,\psi(x) + [E + \alpha(x - x_0)]\psi(x) = E\psi(x)$$

where x_0 is the classical turning point and α is the slope of the potential at that point. Setting $z = (x - x_0)/L$, with the length $L = (\hbar^2/2m\alpha)^{1/3}$, one obtains

$$-\frac{d^2}{dz^2}\,u(z) + zu(z) = 0 \quad \text{with} \quad \psi(x) = u(z)$$

Note that the WKB approximation to this equation yields

$$u(z) \propto z^{-1/4}\exp(\pm z^{3/2}/3) \quad \text{for} \quad z >> 0$$

and

$$u(z) \propto (-z)^{-1/4}\exp(\pm i2(-z)^{3/2}/3) \quad \text{for} \quad z << 0$$

The exact solution to this equation is in terms of the Airy functions:

$$u(z) = \left\{ \begin{array}{c} \mathrm{Ai}(z) \\ \mathrm{Bi}(z) \end{array} \right\}$$

Asymptotic forms of the Airy functions:

$$\mathrm{Ai}(z) \propto \frac{1}{2\sqrt{\pi}\sqrt[4]{z}}\exp(-2z^{3/2}/3) \quad \text{for} \quad |\arg(z)| < \pi \quad \wedge \quad |z| \to \infty \qquad (1)$$

$$\mathrm{Bi}(z) \propto \frac{1}{\sqrt{\pi}\sqrt[4]{z}}\exp(2z^{3/2}/3) \quad \text{for} \quad |\arg(z)| < \pi/3 \quad \wedge \quad |z| \to \infty \qquad (2)$$

and

$$\mathrm{Ai}(-z) \propto \frac{1}{\sqrt{\pi}\sqrt[4]{z}}\sin(2z^{3/2}/3 + \pi/4) \quad \text{for} \quad |\arg(z)| < 2\pi/3 \quad \wedge \quad |z| \to \infty \qquad (3)$$

$$\mathrm{Bi}(-z) \propto \frac{1}{\sqrt{\pi}\sqrt[4]{z}}\cos(2z^{3/2}/3 + \pi/4) \quad \text{for} \quad |\arg(z)| < 2\pi/3 \quad \wedge \quad |z| \to \infty \qquad (4)$$

(1) http://functions.wolfram.com/Bessel-TypeFunctions/AiryAi/06/02/01/01/0001/
(2) http://functions.wolfram.com/Bessel-TypeFunctions/AiryBi/06/02/01/01/0001/
(3) http://functions.wolfram.com/Bessel-TypeFunctions/AiryAi/06/02/01/02/0001/
(4) http://functions.wolfram.com/Bessel-TypeFunctions/AiryBi/06/02/01/02/0001/

For a potential with decreasing slope and $V(x_a) = E$, define

$$I_A = \frac{1}{\hbar}\int_x^{x_a} dx' \sqrt{2m(V(x') - E)} \quad \text{for} \quad x < x_a \;\; i.e. \;\; V(x) > E$$

and

$$I_a = \frac{1}{\hbar}\int_{x_a}^{x} dx' \sqrt{2m(E - V(x'))} \quad \text{for} \quad x > x_a \;\; i.e. \;\; V(x) < E$$

Forms of the matching solutions for $|V(x) - E|^{1/4}\psi(x)$ are then:

$x < x_a$	$x > x_a$
$\exp(-I_A)$ **(conv)**	$\cos(I_a - \pi/4)$
$\exp(I_A)$ **(div)**	$\cos(I_a + \pi/4)$
$\exp(-I_A) \mp i\exp(I_A)$	$e^{\mp i\pi/4}e^{\pm iI_a} = \cos(I_a - \pi/4) \pm i\sin(I_a - \pi/4)$
$e^{\pm i\pi/4}(\exp(-I_A) \mp i\exp(I_A))$	$e^{\pm iI_a} = e^{\pm i\pi/4}(\cos(I_a - \pi/4) \pm i\cos(I_a + \pi/4))$

For a potential with increasing slope and $V(x_b) = E$, define

$$I_b = \frac{1}{\hbar}\int_x^{x_b} dx' \sqrt{2m(E - V(x'))} \quad \text{for} \quad x < x_b \;\; i.e. \;\; V(x) < E$$

and

$$I_B = \frac{1}{\hbar}\int_{x_b}^{x} dx' \sqrt{2m(V(x') - E)} \quad \text{for} \quad x > x_b \;\; i.e. \;\; V(x) > E$$

Forms of the matching solutions for $|V(x) - E|^{1/4}\psi(x)$ are then:

$x < x_b$	$x > x_b$
$\cos(I_b - \pi/4)$	$\exp(-I_B)$ **(conv)**
$\cos(I_b + \pi/4)$	$\exp(I_B)$ **(div)**
$e^{\mp i\pi/4}e^{\pm iI_b} = \cos(I_b - \pi/4) \pm i\sin(I_b - \pi/4)$	$\exp(-I_B) \mp i\exp(I_B)$
$e^{\pm iI_b} = e^{\pm i\pi/4}(\cos(I_b - \pi/4) \pm i\cos(I_b + \pi/4))$	$e^{\pm i\pi/4}(\exp(-I_B) \mp i\exp(I_B))$