

First, 1.13(1) can be written in the form

$$(1) \quad (e^z - 1)^{-1} - z^{-1} + \frac{1}{2} = \sum_{n=1}^{\infty} B_{2n} z^{2n-1} / (2n)! \quad |z| < 2\pi.$$

If the B_{2n} are replaced by 1.13(24) and 1.13(27) we find

$$(2) \quad (e^z - 1)^{-1} = z^{-1} - \frac{1}{2} + 2 \int_0^{\infty} (e^{2\pi t} - 1)^{-1} \sin(tz) dt \quad |\operatorname{Im} z| < 2\pi,$$

$$(3) \quad (e^{2z} - 1)^{-1} = (2z)^{-1} - \frac{1}{2} + \pi z^{-1} \int_0^{\infty} \sin^2(tz) \operatorname{csch}^2(\pi t) dt \quad |\operatorname{Im} z| < \pi.$$

If in 1.13(2) the $B_r(x)$ are replaced by the expressions 1.13(20) and 1.13(21) and in 1.14(2) the $E_r(x)$ by the expressions 1.14(19) and 1.14(20), we find

$$(4) \quad \frac{e^{xz}}{e^z - 1} = \frac{1}{z} + \int_0^{\infty} \frac{\cos(2\pi x) - e^{-2\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \sin(tz) dt \\ - \int_0^{\infty} \frac{\sin(2\pi x)}{\cosh(2\pi t) - \cos(2\pi x)} \cos(tz) dt \quad 0 \leq x < 1, \quad |\operatorname{Im} z| < 2\pi,$$

$$(5) \quad \frac{e^{xz}}{e^z + 1} = 2 \int_0^{\infty} \frac{\sin(\pi x) \cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} \cos(tz) dt \\ - 2 \int_0^{\infty} \frac{\cos(\pi x) \sinh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} \sin(tz) dt \quad 0 \leq x < 1, \quad |\operatorname{Im} z| < \pi.$$

1.16. Polygamma functions

We define

$$(1) \quad \psi^{(n)}(z) = \frac{d^{n+1} \log \Gamma(z)}{dz^{n+1}} = \frac{d^n \psi(z)}{dz^n}, \quad \psi^{(0)}(z) = \psi(z) \quad n = 1, 2, 3, \dots,$$

$$(2) \quad G^{(n)}(z) = \frac{d^n G(z)}{dz^n}, \quad G^{(0)}(z) = G(z) \quad n = 1, 2, 3, \dots$$

The following functional equations are consequences of the results of 1.7.1 and 1.8:

$$(3) \quad \psi^{(n)}(z) - \psi^{(n)}(1+z) = (-1)^{n+1} n! / z^{n+1}$$

$$(4) \quad \psi^{(n)}(z) - (-1)^n \psi^{(n)}(1-z) = -\pi \frac{d^n}{dz^n} [\operatorname{ctn}(\pi z)]$$