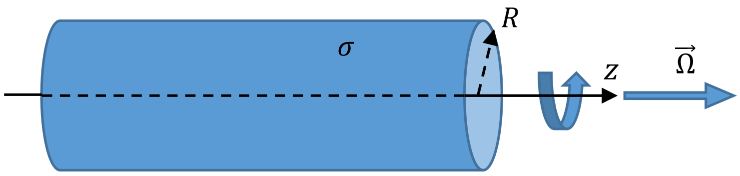
PHYS 102 – General Physics II Final Exam Solution (17.05.2022)

1. An infinite thin cylindrical shell of radius R is uniformly charged with charge density σ on its surface. The cylinder is rotating around its axis, which is taken as the z axis, so that its angular velocity is $\vec{\mathbf{\Omega}} = \Omega_0 \hat{\mathbf{k}}$.

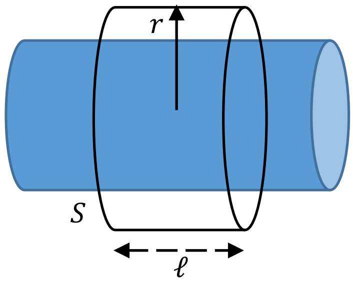
(a) (10 Pts.) Find the expressions for the electric field magnitude as a function of the distance from the z axis, both inside and outside the cylinder.

(b) (10 Pts.) Find the magnetic field **vector** inside and outside the cylinder.



Solution:

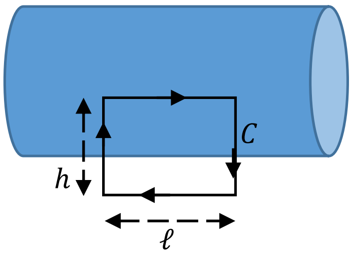
(a) Use Gauss's law applied to the closed cylindrical surface S in the figure.



$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E(2\pi r \ell) = \begin{cases} (2\pi R \ell)\sigma/\epsilon_0 \,, & r > R \\ 0 \,, & r < R \end{cases}$$

$$E(r) = 0 \,, \qquad 0 < r < R \,, \qquad E(r) = \frac{\sigma R}{\epsilon_0 r} \,, \qquad r > R$$

(b) Use Ampère's law applied to the rectangular closed curve C in the figure, noting that the magnetic field is in the $\hat{\mathbf{k}}$-direction (horizontal), and is zero outside the cylinder.



$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}} \quad \rightarrow \quad B\ell = \mu_0 I_{\text{enc}}$$

To find I_{enc} note that the amont of charge $Q = 2\pi R \ell \sigma$ rotates completing one revolution in time $2\pi/\Omega_0$. Therefore

$$I_{\text{enc}} = fQ = \left(\frac{\Omega_0}{2\pi}\right)(2\pi R \ell \sigma) \quad \rightarrow \quad I_{\text{enc}} = R \ell \sigma \Omega_0$$

This means

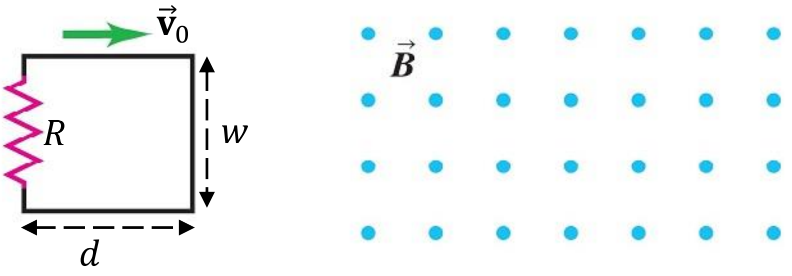
$$B\ell = \mu_0 R \ell \sigma \Omega_0 \quad \rightarrow \quad B = \mu_0 R \sigma \Omega_0 \quad \rightarrow \quad \vec{\mathbf{B}} = \mu_0 R \sigma \Omega_0 \, \hat{\mathbf{k}} \,, \qquad r < R \,, \qquad \vec{\mathbf{B}} = 0 \,, \qquad r > R$$

2. A rectangular circuit is moved at a constant velocity $\vec{\mathbf{v}}_0$ into, through, and then out of a uniform magnetic field $\vec{\mathbf{B}}$, perpendicular to and directed out of the plane of the figure, as shown. The magnetic-field region is considerably wider than w.

(a) (8 Pts.) Find the magnitude and direction (clockwise or counter clockwise) of the current induced in the circuit as it is going into the magnetic field.

(b) (5 Pts.) Find the magnitude and direction of the current induced in the circuit when it is totally within the magnetic field, but still moving.

(c) (7 Pts.) Find the total energy dissipated in the resistance during the whole motion.



Solution:

(a) As the circuit enters the magnetic field region the magnetic flux through it is increasing. By Lenz's law the induced current should be in the **clockwise** direction to create a magnetic field in the opposite direction to the original one opposing the increase.

$$\mathcal{E} = Bwv_0 \quad \rightarrow \quad I_{\text{ind}} = \frac{\mathcal{E}}{R} = \frac{Bwv_0}{R}$$

(b) When the circuit is totally within the magnetic field, but still moving, the magnetic flux through it does not change. Therefore, no emf is induced, and hence, $I_{\text{ind}} = 0$.

(c) As the circuit is entering the magnetic field, power dissipated in the resistor is

$$P = RI_{\text{ind}}^2 \quad \rightarrow \quad P = \frac{B^2w^2v_0^2}{R},$$

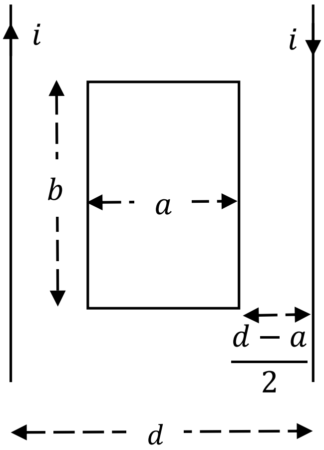
which is constant. Moving with speed v_0, time T it takes for the circuit to completeley enter the magnetic field region is $T = d/v_0$. During this time total power dissipated is

$$W = PT = \left(\frac{B^2w^2v_0^2}{R}\right)\left(\frac{d}{v_0}\right) \quad \rightarrow \quad W = \frac{B^2w^2v_0d}{R}.$$

When the circuit is leaving the magnetic field, the direction of the current will be reversed but, exactly the same amount of power will be dissipated. Therefore, the total power dissipated in the resistance during the whole motion will be

$$W_T = 2W = 2\frac{B^2w^2v_0d}{R}.$$

3. Two parallel infinitely long straight wires separated a distance d are carrying currents i in opposite directions. A rectangular loop of sides a and b, with resistance R, is placed between the wires at an equal distance to both wires, as shown in the figure. The loop and the two wires are on the same plane.

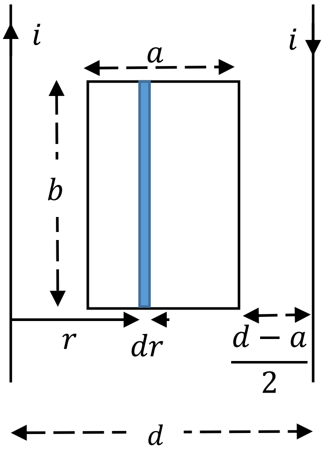


(a) (15 Pts.) Considering the two parallel wires as a single loop of infinite length, find the mutual inductance between the two loops.

(b) (5 Pts.) Find the amplitude of the current induced in the loop if $i = I\sin(\omega t)$.

Solution: (a) Using Ampère's law and the principle of superposition, we see that the magnetic field at a perpendicular distance r from the left wire is directed into and perpendicular to the plane of the rectangle with magnitude

$$B(r) = \frac{\mu_0 i}{2\pi r} + \frac{\mu_0 i}{2\pi(d-r)} = \frac{\mu_0 i}{2\pi}\left(\frac{1}{r} + \frac{1}{d-r}\right).$$



Flux of the magnetic field through the shaded rectangle with height b and width dr is

$$d\Phi = B(r)dA = \frac{\mu_0 i}{2\pi}\left(\frac{1}{r} + \frac{1}{d-r}\right) b\,dr$$

Total flux through the rectangular circuit is

$$\Phi = \int B(r)dA = \frac{\mu_0 ib}{2\pi}\int_{\frac{d-a}{2}}^{\frac{d+a}{2}}\left(\frac{1}{r} + \frac{1}{d-r}\right)dr = \frac{\mu_0 ib}{2\pi}\left[\ln\left(\frac{d+a}{d-a}\right) - \ln\left(\frac{d-a}{d+a}\right)\right] = \frac{\mu_0 ib}{2\pi}\left[\ln\left(\frac{d+a}{d-a}\right)^2\right]$$

$$\Phi = \frac{\mu_0 ib}{\pi}\left[\ln\left(\frac{d+a}{d-a}\right)\right] \quad \rightarrow \quad M = \frac{\Phi}{i} = \frac{\mu_0 b}{\pi}\left[\ln\left(\frac{d+a}{d-a}\right)\right].$$

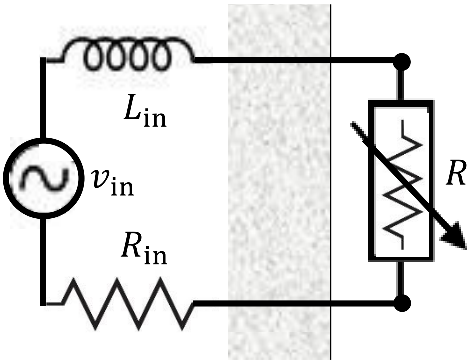
(b)

$$i_{\text{loop}} = \frac{\mathcal{E}}{R}, \qquad |\mathcal{E}| = M\left|\frac{di}{dt}\right| = MI\omega\cos(\omega t) \quad \rightarrow \quad I_{\text{loop}} = \left(\frac{I}{R}\right)\left(\frac{\mu_0 b\omega}{\pi}\right)\left[\ln\left(\frac{d+a}{d-a}\right)\right]$$

4. A space heater is basically an adjustable resistor. We can change its resistance R to adjust the amount of heat it generates. The heater is plugged into a wall outlet. The circuit on the other side of the outlet (inside the wall) can be modelled as an ideal AC voltage source $v_{\text{in}} = V\cos\omega t$, with a series resistance R_{in} and a series inductance L_{in}.

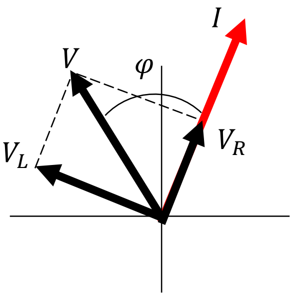
(a) (15 Pts.) What is the value of R for which the power dissipated on it is maximum?

(b) (5 Pts.) If R is adjusted to that value, what is the phase angle between the current in the circuit and the voltage?



Solution:

We have an LR-circuit with resistance $R + R_{\text{in}}$, and inductance L_{in}. The phasor diagram for the circuit is as shown.



$$V_R = I(R + R_{\text{in}})\,, \qquad V_L = I\omega L\,, \qquad V = \sqrt{V_R^2 + V_L^2}$$

$$V = I\sqrt{\omega^2 L^2 + (R + R_{\text{in}})^2} \quad \rightarrow \quad I = V/Z \quad \rightarrow \quad Z = \sqrt{\omega^2 L^2 + (R + R_{\text{in}})^2}$$

$$P_R = \frac{1}{2} R I^2 = \frac{1}{2}\frac{V^2 R}{\omega^2 L^2 + (R + R_{\text{in}})^2}$$

$$\frac{dP_R}{dR} = 0 \quad \rightarrow \quad R = \sqrt{\omega^2 L^2 + R_{\text{in}}^2}$$

(b) Voltage leads the current by an angle φ, where

$$\tan\varphi = \frac{V_L}{V_R} = \frac{\omega L}{R + R_{\text{in}}} \quad \rightarrow \quad \varphi = \arctan\left(\frac{\omega L}{R + R_{\text{in}}}\right)$$

$$R = \sqrt{\omega^2 L^2 + R_{\text{in}}^2} \quad \rightarrow \quad \varphi = \arctan\left(\frac{\omega L}{\sqrt{\omega^2 L^2 + R_{\text{in}}^2} + R_{\text{in}}}\right)$$

5. The average Poynting vector for an electromagnetic wave of wavelength $\lambda = 100\pi$ nm is given as $\vec{\mathbf{S}}_{\text{av}} = 15\ (\hat{\mathbf{i}} + \hat{\mathbf{k}})$.

(a) (3 Pts.) What is the correct SI unit to write at the end of the above formula?

It is also known that the electric field for this electromagnetic wave is only along $\pm\hat{\mathbf{j}}$ direction. Answer the following questions by using the values $c = 3 \times 10^8\ \mathrm{m/s}$, and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}$.

(b) (3 Pts.) What is the angular frequency of this electromagnetic wave?

(c) (3 Pts.) What is the direction of propagation for this electromagnetic wave?

(d) (3 Pts.) What is the maximum value of the electric field magnitude at any point in space?

(e) (3 Pts.) If the Electric field at a point in space is pointing in the $+\hat{\mathbf{j}}$ direction in which direction is the magnetic field vector pointing?

(f) (3 Pts.) What is the ratio of the energy carried by the electric field to the energy carried by the magnetic field?

(g) (2 Pts.) Can your eyes detect this electromagnetic wave?

Solution: (a) $[S] = \mathrm{Watt/m^2}$

(b)

$$\lambda = \pi \times 10^{-7}\ \mathrm{m}, \qquad \omega = ck = 2\pi c/\lambda \quad \rightarrow \quad \omega = 6 \times 10^{15}\ \mathrm{s^{-1}}$$

(c) Direction of the electromagnetic wave is that of the direction of the Poynting vector. i.e., $(\hat{\mathbf{i}} + \hat{\mathbf{k}})$

(d)

$$S = \frac{1}{\mu_0} EB\,, \qquad B = \frac{E}{c} \quad \rightarrow \quad S = \frac{E^2}{\mu_0 c} \quad \rightarrow \quad S_{\text{av}} = \frac{1}{2}\left(\frac{E^2}{\mu_0 c}\right) \quad \rightarrow \quad E = \sqrt{2 S_{\text{av}} \mu_0 c}$$

Given $\vec{\mathbf{S}}_{\text{av}} = 15\ (\hat{\mathbf{i}} + \hat{\mathbf{k}}) \quad \rightarrow \quad S_{\text{av}} = 15\sqrt{2}$

$$E = \sqrt{3600\ \pi\ \sqrt{2}}\ \ \mathrm{N/C}$$

(e) direction of the Poynting vector $\vec{\mathbf{S}}$ is given as $\hat{\mathbf{i}} + \hat{\mathbf{k}}$. Since $\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{E}}$ is in $+\hat{\mathbf{j}}$ direction, direction of the magnetic field must be $-\hat{\mathbf{i}} + \hat{\mathbf{k}}$.

(f)

$$U_E = \epsilon_0 E^2\,, \qquad U_B = B^2/\mu_0\,, \qquad E = cB\,, \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \rightarrow \quad \frac{U_E}{U_B} = 1$$

(g) No! For visible light, 380 nm $< \lambda <$ 750 nm. For the given wave, $\lambda \approx 314$ nm, and is outside the range.