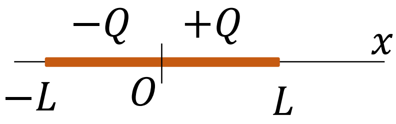
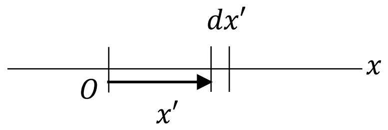
PHYS 102 Final Exam Solution 2021-22-1

1. (20 Pts.) A line charge distribution consists of charge $-Q$ distributed uniformly for $-L < x < 0$, and charge $+Q$ distributed uniformly for $0 < x < +L$. Find the electric potential $V(x)$ and the electric field $\vec{\boldsymbol{E}}(x)$ on the x-axis for $L < x < \infty$.



Solution:



$$dV(x) = \frac{1}{4\pi\epsilon_0}\frac{dq}{x - x'}\,, \qquad dq = \lambda\, dx' \quad \rightarrow \quad dV(x) = \frac{\lambda}{4\pi\epsilon_0}\frac{dx'}{x - x'}\,.$$

For $-L < x < 0$, we have $\lambda = -Q/L$, while for $0 < x < L$, we have $\lambda = Q/L$. Therefore

$$V(x) = \frac{Q}{4\pi\epsilon_0 L}\int_{-L}^{0}\frac{dx'}{x' - x} - \frac{Q}{4\pi\epsilon_0 L}\int_{0}^{L}\frac{dx'}{x' - x} = \frac{Q}{4\pi\epsilon_0 L}\left[\ln\frac{x}{x+L} - \ln\frac{x-L}{x}\right] \quad \rightarrow \quad V(x) = \frac{Q}{4\pi\epsilon_0 L}\ln\left(\frac{x^2}{x^2 - L^2}\right),$$

Using

$$\int \frac{du}{u} = \ln|u|\,,$$

we have

$$V(x) = \frac{Q}{4\pi\epsilon_0 L}\left[\ln\frac{x}{x+L} - \ln\frac{x-L}{x}\right] \quad \rightarrow \quad V(x) = \frac{Q}{4\pi\epsilon_0 L}\ln\left(\frac{x^2}{x^2 - L^2}\right), \qquad L < x < \infty\,.$$

$$\vec{\boldsymbol{E}} = -\frac{dV}{dx}\,\hat{\mathbf{i}} \quad \rightarrow \quad \vec{\boldsymbol{E}} = \frac{QL}{2\pi\epsilon_0 x(x^2 - L^2)}\,\hat{\mathbf{i}}$$

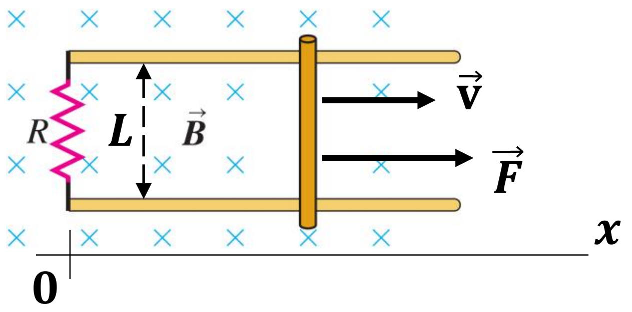
2. Two parallel rails which are L apart are placed along the x direction in a position dependent magnetic field. The magnetic field is directed into the plane of the paper and its magnitude is $B(x) = Ax$, where A is a constant. There is a rod which can slide without friction on the rails and the rails are connected by a resistance R at $x = 0$. All other resistances are negligible. The rod is initially at $x = 0$, but starting at time $t = 0$ it moves with constant velocity $\vec{\mathbf{v}}$ along the x-axis.

(a) (2 Pts.) Find the direction of the current (clockwise or counter clockwise) induced in the circuit.

(b) (6 Pts.) Find the magnitude of the induced current as a function of time.

(c) (6 Pts.) A changing force F must be applied to the rod to keep it moving with constant velocity. Find that force as a function of time.

(d) (6 Pts.) What is the total energy dissipated on the resistor when the rod reaches a distance X?



Solution:

(a) By Lenz's law the induced current should be in the direction opposing the motion. Hence, the magnetic force due to the induced current should be in the opposite direction to the applied force $\vec{F}$. This means the induced current should be counterclockwise.

(b) Motional emf with constant velocity $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$. $x = vt \quad \to \quad B(x) = Avt$.

$$\mathcal{E} = B(x)Lv = Av^2Lt \quad \to \quad I_{\text{ind}} = \frac{\mathcal{E}}{R} = \frac{Av^2L}{R}t\,.$$

(c) To maintain constant velocity, the applied force must be equal in magnitude to the magnetic force due to the induced current.

$$F = F_M = B(x)I_{\text{ind}}L = (Avt)\left(\frac{Av^2L}{R}t\right)L = \frac{1}{R}A^2L^2v^3t^2\,.$$

Therefore

$$\vec{F} = \left(\frac{1}{R}A^2L^2v^3t^2\right)\hat{\mathbf{i}}\,.$$

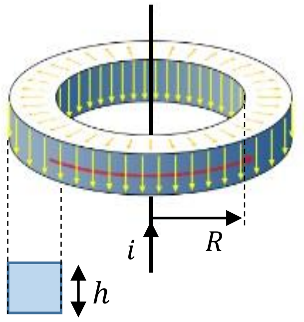
(d) Power dissipated by the resistor is $P = RI_{\text{ind}}^2$. Hence, the total energy dissipated is

$$E = \int_0^T P(t)\,dt = \int_0^T R\left(\frac{Av^2L}{R}t\right)^2 dt \quad \to \quad E = \frac{1}{R}A^2v^4L^2\int_0^T t^2\,dt = \frac{1}{3R}A^2v^4L^2T^3\,.$$

Since time required to reach the distance X is $T = X/v$, The result becomes

$$E = \frac{1}{3R}A^2vL^2X^3\,.$$

3. An infinitely long straight wire carrying a current $i(t) = I\cos(\omega t)$ is situated on the symmetry axis of a toroid with tightly wound N turns. The toroid has an inner radius R and a square cross section with dimension h.



(a) (5 Pts.) Find the magnitude of the magnetic field created by the current as a function of the perpendicular distance r from the wire .

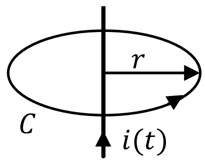
(b) (5 Pts.) Find the magnetic flux through one turn of the toroid.

(c) (5 Pts.) Find the mutual induction between the toroid and the infinite wire.

(d) (5 Pts.) Find the emf induced on the toroid by the current in the wire.

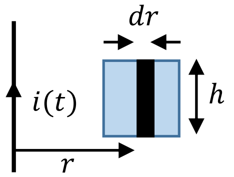
Solution:

(a) Using Ampère's law $\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}}$, where C is a circle of radius r whose symmetry axis coincides with the current carrying wire, we find the magnitude of the magnetic field created by the current as



$$B(r) = \frac{\mu_0 i(t)}{2\pi r}.$$

(b) Magnetic flux through one turn of the toroid is equal to the flux through a square cross section of dimension h.



$$\Phi_B = \int d\Phi = \int_R^{R+h} B(r)\, h dr = \frac{\mu_0 i(t) h}{2\pi} \int_R^{R+h} \frac{dr}{r} \quad \rightarrow \quad \Phi_B = \frac{\mu_0 i(t) h}{2\pi} \ln\left(\frac{R+h}{R}\right).$$

(c) Total magnetic flux through the toroid is equal to the total number of turns N multiplied by the flux through a square cross section.

$$\Phi_{\text{tot}} = N\Phi_B = \frac{\mu_0 N i(t) h}{2\pi} \ln\left(\frac{R+h}{R}\right).$$

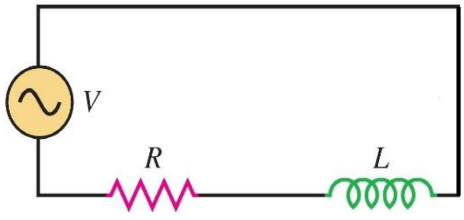
Since mutual inductance is defined as $M = \Phi_{\text{tot}}/i(t)$, we find

$$M = \frac{\mu_0 N h}{2\pi} \ln\left(\frac{R+h}{R}\right).$$

(d)

$$\mathcal{E} = M\frac{di}{dt} = -\frac{\mu_0 N h}{2\pi} \ln\left(\frac{R+h}{R}\right) \omega I \sin(\omega t)\,.$$

4. A resistor R and an inductor L are connected in series to an AC voltage source $V(t) = V_0 \cos(\omega t)$. Assume that the circuit has been running for a long time.



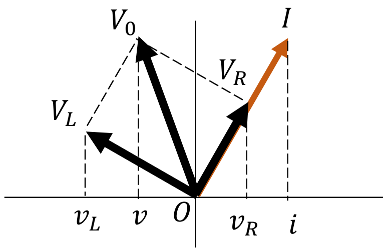
(a) (5 Pts.) Find the maximum value of the current running through the circuit.
(b) (5 Pts.) Find the average power dissipated on the resistor.

At a time when the current in the circuit is at its maximum the voltage source breaks down and is not able to supply any voltage. After that time the voltage source is effectively replaced by just a resistor R_S.

(c) (5 Pts.) Find how long it would take after the breakdown for the current in the circuit to fall to half of its value.

(d) (5 Pts.) Find the total energy dissipated on the resistor R after the breakdown.

Solution: (a) Voltage current relations of an RL circuit can be describd by the phasor diagram shown. We have $v = V_0 \cos \omega t$ and $i = I \cos(\omega t + \varphi)$, so the maximum value of the current is denoted by I. We also know that



$$V_R = IR\,, \qquad V_L = I\omega L \quad \rightarrow \quad V_0 = I\sqrt{R^2 + \omega^2 L^2}\,.$$

Therefore,

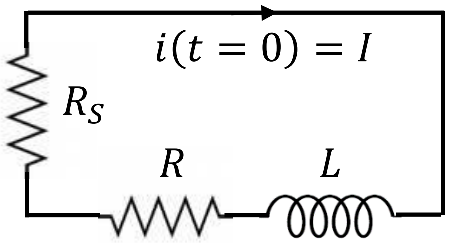
$$I = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\,.$$

(b) Average power dissipated on the resistor is

$$P_{\text{av}} = V_{\text{Rrms}} I_{\text{rms}} = \frac{V_R}{\sqrt{2}} \frac{I}{\sqrt{2}} = \frac{1}{2} R I^2 \quad \rightarrow \quad P_{\text{av}} = \frac{R V_0^2}{2(R^2 + \omega^2 L^2)}\,.$$

(c) When $(i = I)$ the source breaks down, the circuit is replaced by one with two resistors and an inductor connected in series. Assume that this happens at time $t = 0$. Writing the loop rule, we have the following initial value problem.

$$L\frac{di}{dt} + (R + R_S)i = 0\,, \qquad i(0) = I\,.$$



The solution is

$$i(t) = I e^{-(R+R_S)t/L}\,.$$

Therefore, if $i(T) = I/2$, we have

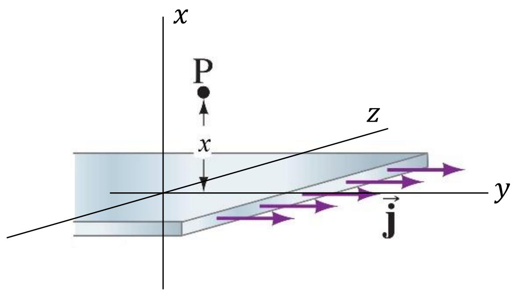
$$\frac{i}{I} = \frac{1}{2} = e^{-(R+R_S)T/L} \quad \rightarrow \quad T = \frac{L}{R + R_S} \ln 2\,.$$

(d) Instantaneous power dissipated by the resistor R is $P(t) = i^2 R = I^2 R e^{-2(R+R_S)t/L}$. The total energy dissipated is

$$U_R = \int_0^\infty P(t)dt = I^2 R \int_0^\infty e^{-2(R+R_S)t/L} dt \quad \rightarrow \quad U_R = \frac{RL}{2(R + R_S)} \frac{V_0^2}{(R^2 + \omega^2 L^2)}\,.$$

Note: One can verify that the total energy dissipated by the two resistors is $U_R + U_{R_S} = \frac{1}{2} L I^2$, which is equal to the total energy stored in the inductor at time $t = 0$.

5. A very large flat conducting sheet of thickness w carries a uniform current density $\vec{\mathbf{J}} = \mathrm{J}\,\hat{\mathbf{j}}$ throughout.



(a) (7 Pts.) Determine the magnetic field $\vec{\boldsymbol{B}}$ at a distance x above the plane. (Assume the plane is infinitely long and wide.)

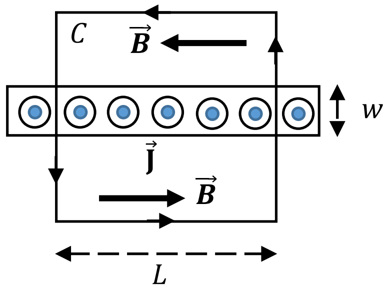
If the current in the y-direction oscillates in time according to the expression $\vec{\mathbf{J}} = \mathrm{J}\cos(\omega t)\ \hat{\mathbf{j}}$, the sheet radiates an electromagnetic wave propagating in the $\pm x$-directions. Such electromagnetic waves are emitted from all points on the sheet. The magnetic field component of the wave propagating in the $+x$-direction is described by the wave function $\vec{\boldsymbol{B}} = B[\cos(kx - \omega t)](-\hat{\mathbf{k}})$, where B is found in part (a).

(b) (7 Pts.) Find the wave function for the electric field component of the wave propagating in the $+x$-direction.

(c) (6 Pts.) Find the Poynting vector for $x > 0$ as a function of x and t.

Solution:

(a) Because of the symmetry of the problem, magnetic field on the top of the sheet ($x > 0$) will be in the $-\hat{\mathbf{k}}$ direction, while magnetic field on the bottom of the sheet ($x < 0$) will be in the $\hat{\mathbf{k}}$ direction. Applying Ampère's law $\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}}$ to the closed curve C shown, we have



$$2BL = \mu_0 L w \mathrm{J} \quad \rightarrow \quad B = \frac{1}{2}\mu_0 w \mathrm{J} \quad \rightarrow \quad \vec{\boldsymbol{B}} = \frac{1}{2}\mu_0 w \mathrm{J}\,(-\hat{\mathbf{k}})\,.$$

(b) Direction of propagation of the wave is $\hat{\mathbf{i}}$. This means $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ is in the $\hat{\mathbf{i}}$ direction. Since $(-\hat{\mathbf{j}}) \times (-\hat{\mathbf{k}}) = \hat{\mathbf{i}}$, we have $\vec{\boldsymbol{E}} = E[\cos(kx - \omega t)](-\hat{\mathbf{j}})$. We also have $E = Bc$, where c is the speed of light in vacuum. Therefore

$$\vec{\boldsymbol{E}} = Bc[\cos(kx - \omega t)](-\hat{\mathbf{j}}) \quad \rightarrow \quad \vec{\boldsymbol{E}} = \frac{1}{2}\mu_0 w \mathrm{J} c[\cos(kx - \omega t)](-\hat{\mathbf{j}})\,.$$

(c)

$$\vec{S} = \frac{1}{\mu_0}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \frac{1}{\mu_0}B^2 c[\cos(kx - \omega t)]^2\hat{\mathbf{i}} \quad \rightarrow \quad \vec{S} = \frac{1}{4}\mu_0 w^2 \mathrm{J}^2 c[\cos(kx - \omega t)]^2\hat{\mathbf{i}}\,.$$