PHYS 102 Final Exam Solution 2020-21-2

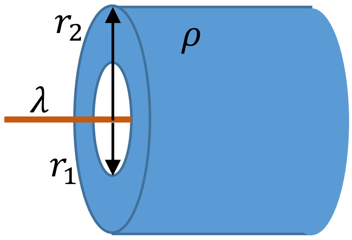
PHYS 102 Final Exam Question 1

A very long line charge with uniform density λ is surrounded by a concentric cylindrical tube of inner radius r_1 and outer radius r_2, containing uniform volume charge density ρ. The electric field outside the tube ($r > r_2$) is zero.

(a) (5 Pts.) Find the relation between λ and ρ.

(b) (10 Pts.) Find the electric field magnitude for $r < r_1$, and $r_1 < r < r_2$. Express your result in terms of ρ.

(c) (10 Pts.) Find the potential difference $V(r_1) - V(r_2)$. Express your result in terms of ρ.



Solution: (a) If the electric field outside the tube is zero, the net charge of a section of unit length must be zero.

$$\lambda + \pi(r_2^2 - r_1^2)\rho = 0\,.$$

(b) We consider a Cylindrical Gauss surface with length L, radius $r < r_1$, and axis on the line charge.

$$\oint \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}\, dA = 2\pi r L\, E(r) = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{\lambda}{2\pi\epsilon_0 r} = -\frac{(r_2^2 - r_1^2)\rho}{2\epsilon_0 r}\,, \qquad 0 < r < r_1\,.$$

Now consider a Cylindrical Gauss surface with length L, radius $r_1 < r < r_2$, and axis on the line charge.

$$\oint \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}}\, dA = 2\pi r L\, E(r) = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{L}{\epsilon_0}[\lambda + \pi(r^2 - r_1^2)\rho] \quad \rightarrow \quad E(r) = \frac{\lambda + \pi(r^2 - r_1^2)\rho}{2\pi\epsilon_0 r}\,.$$

Hence,

$$E(r) = \frac{\rho}{2\epsilon_0}\left(r - \frac{r_2^2}{r}\right) = \frac{\lambda}{2\pi\epsilon_0 r}\left(\frac{r_2^2 - r^2}{r_2^2 - r_1^2}\right).$$

(c)

$$V(r_1) - V(r_2) = \int_{r_1}^{r_2} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = \frac{\rho}{2\epsilon_0}\int_{r_1}^{r_2}\left(r - \frac{r_2^2}{r}\right) dr = \frac{\rho}{2\epsilon_0}\left(\frac{r_2^2}{2} - \frac{r_1^2}{2} - r_2^2 \ln\frac{r_2}{r_1}\right).$$

Same expression in terms of λ is

$$V(r_1) - V(r_2) = \frac{-\lambda}{2\epsilon_0\pi(r_2^2 - r_1^2)}\left(\frac{r_2^2}{2} - \frac{r_1^2}{2} - r_2^2 \ln\frac{r_2}{r_1}\right).$$

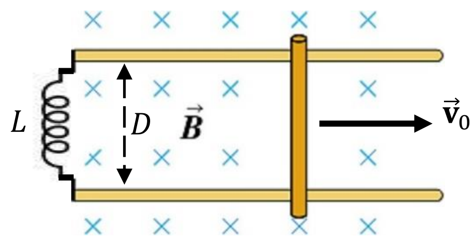
PHYS 102 Final Exam Question 2

The figure shows a metal bar of mass m, as seen from above, placed on two horizontal parallel conducting rails a distance D apart. The circuit is completed at one end of the parallel rails by a coil of inductance L. The resistance of the whole circuit can be ignored. A uniform magnetic field with magnitude B that is directed into the plane of the figure exists in the whole region. The bar, which initially was at rest, is given an initial velocity $\vec{\mathbf{v}}_0$ at time $t = 0$.

(a) (15 Pts.) Find the current through the coil as a function of time, noting that $i(t = 0) = 0$.

(b) (5 Pts.) Find the velocity of the wire as a function of time.

(c) (5 Pts.) Find the maximum energy stored in the coil and comment on the result.



Solution: (a) If $v(t)$ is the speed of the bar at time t, then $\mathcal{E}(t) = BDv(t)$ is the motional emf induced at the two ends of the rod. This is also equal to the voltage across the inductor. So

$$\mathcal{E} = L\frac{di}{dt} = BDv \quad \rightarrow \quad L\frac{d^2i}{dt^2} = BD\frac{dv}{dt}.$$

Magnetic force on the bar will be in the opposite direction to the velocity (Lenz's law). Therefore,

$$F_B = BDi \quad \rightarrow \quad m\frac{dv}{dt} = -BDi \quad \rightarrow \quad L\frac{d^2i}{dt^2} = BD\frac{dv}{dt} = -\frac{B^2D^2}{m}i\,.$$

Hence, the current through the inductor satisfies the differential equation

$$\frac{d^2i}{dt^2} + \frac{B^2D^2}{mL}i = 0\,.$$

Solution of this equation satisfying the condition $i(0) = 0$ is

$$i(t) = I_{\max}\sin\left(\frac{BD}{\sqrt{mL}}\,t\right).$$

(b) We had

$$BDv(t) = L\frac{di}{dt} \quad \rightarrow \quad v(t) = \sqrt{\frac{L}{m}}I_{\max}\cos\left(\frac{BD}{\sqrt{mL}}\,t\right).$$

For $t = 0$

$$v(0) = \sqrt{\frac{L}{m}}I_{\max} = v_0 \quad \rightarrow \quad I_{\max} = v_0\sqrt{\frac{m}{L}} \quad \rightarrow \quad i = v_0\sqrt{\frac{m}{L}}\,\sin\left(\frac{BD}{\sqrt{mL}}\,t\right), \qquad v = v_0\,\cos\left(\frac{BD}{\sqrt{mL}}\,t\right).$$

(c)

$$U_L = \frac{1}{2}Li^2 = \frac{1}{2}mv_0^2\,\left(\sin\left(\frac{BD}{\sqrt{mL}}\,t\right)\right)^2 \quad \rightarrow \quad U_{L\,\max} = \frac{1}{2}mv_0^2.$$

Maximum energy stored in the coil is equal to the initial kinetic energy of the bar.

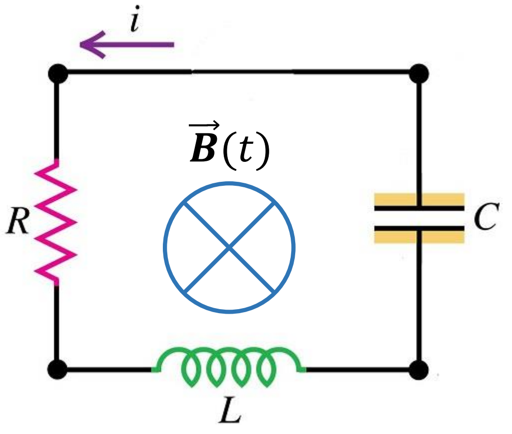
PHYS 102 Final Exam Question 3

A wire loop connects a capacitor C, an inductor L, and a resistor R in series, and has a total area A. The loop is placed in a perpendicular magnetic field with time dependent amplitude. $B(t) = B\sin(\omega t)$. Assume that the magnetic field has been oscillating for a long time and the system has reached steady state.

(a) (9 Pts.) Find the time dependent current in the loop, $i(t) = I\cos(\omega t + \varphi)$ by calculating both I and φ.

(b) (8 Pts.) What is the average power dissipated on the resistor?

(c) (8 Pts.) What is the maximum energy stored in the capacitor?



Solution: We use Faraday's law and Kirchhoff rule.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B(t)A) = -BA\omega\cos(\omega t)\text{, and } \mathcal{E} = v_R + v_L + v_C\,.$$

$$v_R = i(t)R = IR\cos(\omega t + \varphi)\ , \qquad v_L = L\frac{di}{dt} = -\omega L I\sin(\omega t + \varphi)$$

and $v_C = \frac{q}{C} = \frac{1}{C}\int i(t)dt = \frac{I}{\omega C}\sin(\omega t + \varphi)$, means that we have

$$-BA\omega\cos(\omega t) = IR\cos(\omega t + \varphi) - \omega L I\sin(\omega t + \varphi) + \frac{I}{\omega C}\sin(\omega t + \varphi).$$

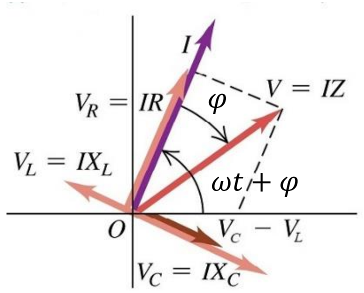
Writing $\cos(\omega t) = \cos(\omega t + \varphi - \varphi) = \cos(\omega t + \varphi)\cos(\varphi) + \sin(\omega t + \varphi)\sin(\varphi)$, the equation becomes

$$[BA\omega\ \cos(\varphi) + IR]\cos(\omega t + \varphi) + \left[BA\omega\ \sin(\varphi) - \left(\omega L - \frac{1}{\omega C}\right)I\right]\sin(\omega t + \varphi) = 0\,.$$

If this equation is satisfied for all values of t, coefficients of the functions $\cos(\omega t + \varphi)$ and $\sin(\omega t + \varphi)$ must be zero.

$$BA\omega\ \cos(\varphi) + IR = 0, BA\omega\ \sin(\varphi) - \left(\omega L - \frac{1}{\omega C}\right)I = 0 \quad \to \quad I = \frac{BA\omega}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\ , \qquad \tan\varphi = \left(\frac{1}{\omega RC} - \frac{\omega L}{R}\right).$$

One can obtain the same result using the phasor diagram.



$$V^2 = B^2A^2\omega^2 = V_R^2 + (V_L - V_C)^2 = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]I^2\ , \qquad \tan(\varphi) = \frac{V_C - V_L}{V_R}.$$

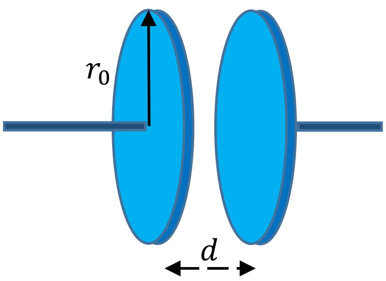
(b)

$$P_{R\,\text{av}} = \frac{1}{2}I^2R = \frac{1}{2}\frac{B^2A^2\omega^2R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

(c)

$$U_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}\frac{I^2}{\omega^2C}(\sin(\omega t + \varphi))^2 \quad \to \quad U_{C\,\max} = \frac{1}{2}\frac{I^2}{\omega^2C} = \frac{1}{2C}\frac{B^2A^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

PHYS 102 Final Exam Question 4



An air gap capacitor with circular plates of radius r_0 and plate separation d is connected to an AC source whose voltage is given by $v(t) = V_0 \cos \omega t$. Answer the following in terms of the given parameters r_0, d, V_0, ω, and the necessary constants.

(a) (5 Pts.) What is the current in the capacitor as a function of time?

(b) (5 Pts.) What is the electric field magnitude in the gap between the plates?

(c) (5 Pts.) What is the magnetic field magnitude in the gap between the plates as a function of the distance r from the symmetry axis?

(d) (5 Pts.) What is the magnitude of the Poynting vector as a function of the distance r from the symmetry axis?

(e) (5 Pts.) What is the expression for the energy stored in the capacitor?

Solution: (a) By definition, charge on the capacitor is $q = Cv$. Since for a parallel plate capacitor $C = \epsilon_0 A/d$,

$$i = \frac{dq}{dt} = C\frac{dv}{dt} = -\epsilon_0 \frac{\pi r_0^2}{d} \omega V_0 \sin \omega t\,.$$

(b)

$$|E| = \frac{|v(t)|}{d} = \frac{V_0}{d}|\cos \omega t|\,.$$

(c) Applying the Ampère-Maxwell law on a circle centered at the axis of symmetry with radius $r < r_0$, we get

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad \rightarrow \quad 2\pi r B = -\mu_0 \epsilon_0 \pi r^2 \frac{V_0}{d} \omega \sin \omega t \quad \rightarrow \quad |B| = \frac{\mu_0 \epsilon_0 r V_0 \omega}{2d} |\sin \omega t|\,.$$

(d) Since the vectors $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are perpendicular to each other,

$$\vec{\boldsymbol{S}} = \frac{1}{\mu_0} \vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} \quad \rightarrow \quad |\vec{\boldsymbol{S}}| = \frac{1}{\mu_0} |\vec{\boldsymbol{E}}||\vec{\boldsymbol{B}}| = \frac{\epsilon_0 r V_0^2 \omega}{4d^2} |\sin(2\omega t)|\,.$$

(e)

$$U_C = \frac{1}{2} C v^2 = \epsilon_0 \frac{\pi r_0^2}{2d} V_0^2 (\cos \omega t)^2\,.$$