# PHYS 102 Final Exam

**1.** A particle with charge $q$ is moving in a uniform magnetic field $\vec{\boldsymbol{B}} = B\,\hat{\mathbf{k}}$. The magnetic force on the particle is measured to be $\vec{\boldsymbol{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$.

(a) (13 Pts.) Calculate the velocity components of the particle that you can from this information.

(b) (6 Pts.) Are there components of the velocity that are not determined by the measurement of the force? Explain.

(c) (6 Pts.) What is the angle between $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{F}}$?

**Solution:**

(a) The magnetic force on a particle is given by

$$\vec{\boldsymbol{F}} = q\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} = q\left(v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}\right) \times \left(B\,\hat{\mathbf{k}}\right) = qv_yB\,\hat{\mathbf{i}} - qv_xB\,\hat{\mathbf{j}}\,.$$

Therefore,

$$qv_yB\,\hat{\mathbf{i}} - qv_xB\,\hat{\mathbf{j}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} \quad \rightarrow \quad v_x = -\frac{F_y}{qB}\,, \qquad v_y = \frac{F_x}{qB}\,.$$

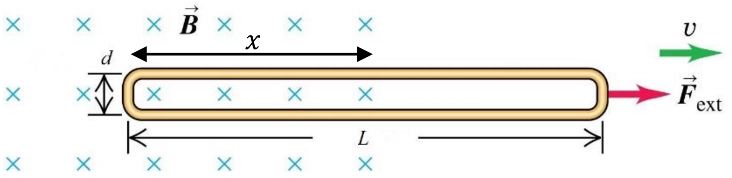
(b) $z$-component of the velocity, $v_z$ can not be determined from the given information because $v_z\hat{\mathbf{k}} \times B\hat{\mathbf{k}} \equiv 0$ and, for this reason, $v_z$ does not appear in the expression for the force.

(c) Since

$$\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{F}} = \left(F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}\right) \cdot \left(B\,\hat{\mathbf{k}}\right) = 0\,,$$

the magnetic field is perpendicular to the force. The smaller of the angles between the two vectors is $\pi/2$.

**2.** A rectangular loop of wire can slide without friction on a horizontal surface. Initially the loop has part of its area in a region of uniform magnetic field that has magnitude $B$ and is perpendicular to the plane of the loop. The loop has dimensions $L$ by $d$, mass $m$, and resistance $R$. The loop is initially at rest. At time $t = 0$, a constant force of magnitude $F_{\text{ext}}$ is applied to the loop to pull it out of the field as shown in the figure.




(a) (10 Pts.) Find the speed of the loop as a function of time before it leaves the magnetic field.

(b) (10 Pts.) Find the power dissipated in the loop as a function of time before it leaves the magnetic field.

(c) (5 Pts.) Find the power supplied by the external force as a function of time before the loop leaves the magnetic field.

**Solution:**

(a) Flux of the magnetic field through the loop as a funciton of $x$ is $\Phi_B = Bxd$. Therefore, when the loop starts to move under the action of the external force, there will be an induced emf causing an induced current.

$$|\mathcal{E}_{\text{ind}}| = \frac{d\Phi_B}{dt} = B\frac{dx}{dt}d = Bvd \quad \rightarrow \quad I_{\text{ind}} = \frac{|\mathcal{E}_{\text{ind}}|}{R} = \frac{Bvd}{R}\,.$$

This induced current will result in a magnetic force acting on the loop which, by Lenz's law, will be in the opposite direction of the external force causing the change in the magnetic flux. Net force on the loop is found as

$$F_B = I_{\text{ind}}Bd = \frac{B^2d^2}{R}v \quad \rightarrow \quad F_{\text{net}} = F_{\text{ext}} - \frac{B^2d^2}{R}v\,.$$

Writing Newton's second law, we obtain

$$m\frac{dv}{dt} = F_{\text{ext}} - \frac{B^2d^2}{R}v \quad \rightarrow \quad \int_0^v \frac{dv'}{\left(v' - \frac{RF_{\text{ext}}}{B^2d^2}\right)} = -\frac{B^2d^2}{mR}\int_0^t dt'\,,$$

which gives

$$\ln\frac{v - \frac{F_{\text{ext}}R}{B^2d^2}}{\frac{F_{\text{ext}}R}{B^2d^2}} = -\frac{B^2d^2}{mR}t \quad \rightarrow \quad v(t) = \frac{F_{\text{ext}}R}{B^2d^2}\left(1 - e^{-\frac{B^2d^2}{mR}t}\right).$$

(b)

$$P_{\text{dis}} = I_{\text{ind}}^2R = \frac{B^2d^2}{R}v^2 \quad \rightarrow \quad P_{\text{dis}} = \frac{F_{\text{ext}}^2R}{B^2d^2}\left(1 - e^{-\frac{B^2d^2}{mR}t}\right)^2.$$

(c)

$$P_{\text{sup}} = \vec{F}_{\text{ext}}\cdot\vec{v} = \frac{F_{\text{ext}}^2R}{B^2d^2}\left(1 - e^{-\frac{B^2d^2}{mR}t}\right).$$

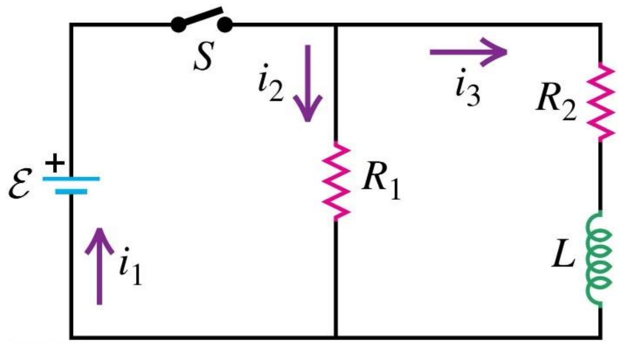
Note that in the limit $t \rightarrow \infty$ (at terminal speed) $P_{\text{dis}} = P_{\text{sup}}$.

**3.** An inductor with inductance $L$ and negligible resistance is connected to a battery, a switch $S$, and two resistors, $R_1$ and $R_2$ as shown in the figure. The battery has emf $\mathcal{E}$ and negligible internal resistance. $S$ is closed at $t = 0$.

(a) (7 Pts.) What are the currents $i_1$, $i_2$, and $i_3$ just after $S$ is closed?

(b) (7 Pts.) What are $i_1$, $i_2$, and $i_3$ after S has been closed a long time?

(c) (11 Pts.) If the battery is replaced by an AC voltage source with $v(t) = V\cos\omega t$, what are the current amplitudes $I_2$, and $I_3$?




**Solution:**

(a) Just after the switch is closed the inductor behaves like an open circuit, hence no current passes through it. So

$$i_1 = i_2 = \frac{\mathcal{E}}{R_1}\,, \qquad i_3 = 0\,.$$

(b) After S has been closed a long time the inductor behaves like a short circuit and there is no potential drop across it. We then have two resistors connected in parallel with equivalent resistance $R_\mathrm{e}$, and with potential difference $\mathcal{E}$. So

$$i_1 = i_2 + i_3\,, \qquad i_2 = \frac{\mathcal{E}}{R_1}\,, \qquad i_3 = \frac{\mathcal{E}}{R_2} \quad \rightarrow \quad i_1 = \frac{\mathcal{E}}{R_1} + \frac{\mathcal{E}}{R_2} = \mathcal{E}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{\mathcal{E}}{R_e}\,.$$

(c) If the battery is replaced by an AC voltage source with $v(t) = V\cos\omega t$, we have

$$i_2 = \frac{v_R}{R_1} = \frac{V}{R_1}\cos\omega t \quad \rightarrow \quad I_2 = \frac{V}{R_1}\,,$$

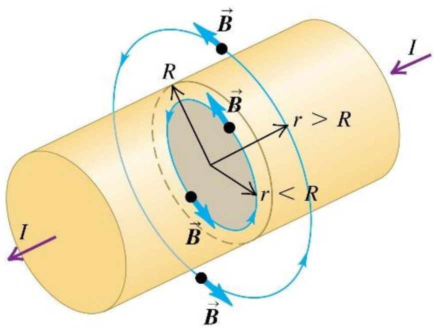
in phase with the voltage. Let $i_3 = I_3\cos(\omega t + \phi)$ passing through $R_2$ and the inductor. Voltage $v_{R2}$ on $R_2$ is $v_{R2} = i_3 R_2 = I_3 R_2 \cos(\omega t + \phi)$ in phase with the current, while the voltage on the inductor is

$$v_L = L\frac{di_3}{dt} = -L\omega I_3 \sin(\omega t + \phi)$$

leading the current by $90^\circ$. This means

$$V = I_3\sqrt{R_2^2 + L^2\omega^2} \quad \rightarrow \quad I_3 = \frac{V}{\sqrt{R_2^2 + L^2\omega^2}}\,.$$

**4.** A cylindrical conductor with radius $R$ and resistivity $\rho$ carries a current $I$. The current is uniformly distributed over the cross-sectional area of the conductor.




(a) (10 Pts.) Find the magnitude of the magnetic field as a function of the distance r from the conductor axis for points both inside ($0 < r < R$) and outside ($R < r < \infty$) the conductor if the current is constant in time.

(b) (5 Pts.) What is the electric field inside the conductor?

(c) (10 Pts.) Find the magnitude of the magnetic field as a function of the distance r from the conductor axis for points both inside ($0 < r < R$) and outside ($R < r < \infty$) the conductor if the current decreases at a constant rate $\beta$ (i.e., $I = I_0 - \beta t$).

**Solution:**

(a) We use Ampère's law where the integral is taken around the circle with radius $r$.

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}} \quad \rightarrow \quad B(r)(2\pi r) = \mu_0 I_{\text{enc}} \quad \rightarrow \quad B(r) = \frac{\mu_0 I_{\text{enc}}}{2\pi r}.$$

For $0 < r < R$,

$$I_{\text{enc}} = JA = \left(\frac{I}{\pi R^2}\right)(\pi r^2) = \frac{I r^2}{R^2} \quad \rightarrow \quad B(r) = \frac{\mu_0 I r}{2\pi R^2}, \qquad 0 < r < R.$$

For $R < r < \infty$,

$$I_{\text{enc}} = I \quad \rightarrow \quad B(r) = \frac{\mu_0 I}{2\pi r}, \qquad R < r < \infty.$$

(b) By definition of resistivity, electric field inside the conductor is

$$\rho = \frac{E}{J} \quad \rightarrow \quad E = \rho J = \frac{\rho I}{\pi R^2}.$$

(c) If the current changes in time, we also have a displacement current. Therefore, we must use Ampère-Maxwell law.

$$\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \left(I_c + \epsilon_0 \frac{d\Phi_E}{dt}\right).$$

For $0 < r < R$,

$$\Phi_E = EA = \left(\frac{\rho I}{\pi R^2}\right)(\pi r^2) = \frac{\rho I r^2}{R^2} \quad \rightarrow \quad \frac{d\Phi_E}{dt} = \frac{\rho r^2}{R^2}\frac{dI}{dt} = -\frac{\beta \rho r^2}{R^2}.$$

Therefore

$$B(r)(2\pi r) = \mu_0 \left(\frac{I r^2}{R^2} - \epsilon_0 \frac{\beta \rho r^2}{R^2}\right) \quad \rightarrow \quad B(r) = \frac{\mu_0 r}{2\pi R^2}(I - \epsilon_0 \beta \rho), \qquad 0 < r < R.$$

For $R < r < \infty$,

$$\Phi_E = EA = \left(\frac{\rho I}{\pi R^2}\right)(\pi R^2) = \rho I \quad \rightarrow \quad \frac{d\Phi_E}{dt} = \rho \frac{dI}{dt} = -\beta \rho.$$

Therefore,

$$B(r)(2\pi r) = \mu_0 (I - \epsilon_0 \beta \rho) \quad \rightarrow \quad B(r) = \frac{\mu_0 (I - \epsilon_0 \beta \rho)}{2\pi r}, \qquad R < r < \infty.$$