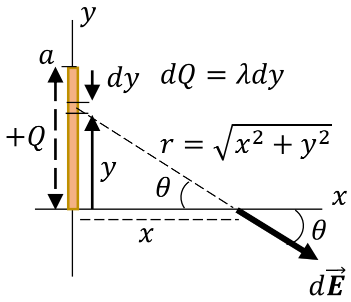
PHYS 102 – General Physics II Midterm Exam Solution

1. (25 Pts.) Charge $+Q$ is distributed unifomly along the positive y-axis for $0 < y < a$, and charge $-Q$ is distributed uniformly along the negative y-axis for $-a < y < 0$ as shown in the figure.

(a) (5 Pts.) What is the direction of the electric field on the x-axis?

(b) (20 Pts.) Find the magnitude $|\vec{E}(x)|$ of the electric field on the x-axis for $x > 0$.

Solution:



Let us first consider the $+Q$ charge distributed uniformly along the positive y-axis for $0 < y < a$. From the figure

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

According to Coulomb's law, electric field produced at the point $(x, 0)$ by the infinitesimal charge $dQ = \lambda dy = \frac{Q}{a} dy$ is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2+y^2}(\cos\theta\ \hat{\imath} - \sin\theta\ \hat{\jmath}) = \frac{Q}{4\pi\epsilon_0 a}\left(\frac{x\,\hat{\imath} - y\,\hat{\jmath}}{(x^2+y^2)^{3/2}}\right)dy$$

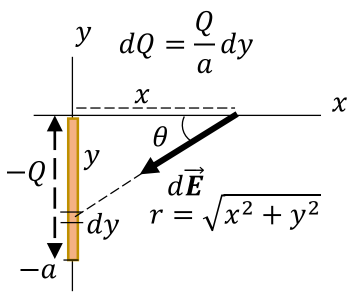
Integrating over y from $y = -a$ to $y = a$, we find

$$E_x = \frac{Qx}{4\pi\epsilon_0 a}\int_0^a \frac{dy}{(x^2+y^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0 x\sqrt{x^2+a^2}}$$

and

$$E_y = \frac{-Q}{4\pi\epsilon_0 a}\int_0^a \frac{y\,dy}{(x^2+y^2)^{3/2}} = \frac{-Q}{4\pi\epsilon_0 a}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right)$$

Let us now consider the $-Q$ charge distributed uniformly along the negative y-axis for $-a < y < 0$. Considering the symmetry of the problem, we can write



$$E_x = \frac{-Q}{4\pi\epsilon_0 x\sqrt{x^2+a^2}}$$

and

$$E_y = \frac{-Q}{4\pi\epsilon_0 a}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right)$$

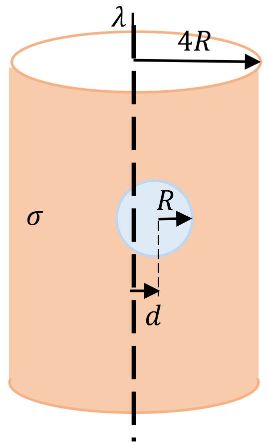
The total electric field is

$$\vec{E} = \frac{-Q}{2\pi\epsilon_0 a}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right)\hat{\jmath}$$

(a) By symmetry direction is $-\hat{\jmath}$. (b)

$$|\vec{E}| = \frac{Q}{2\pi\epsilon_0 a}\left(\frac{1}{x} - \frac{1}{\sqrt{x^2+a^2}}\right).$$

2. A very long and **thin cylindrical shell** of radius $4R$ has a uniform surface charge density σ. It is concentric with a **line of charge** with uniform charge density λ.



(a) (8 Pts.) Find the magnitude of the electric field for $0 < r < 4R$ and $4R < r < \infty$, where r is the perpendicular distance from the line charge.

(b) (10 Pts.) Find the potential differences $V(3R) - V(R)$ and $V(5R) - V(R)$.

(c) (7 Pts.) Find the flux $\oint \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dA}}$ of the electric field through the surface of a sphere of radius R placed inside the shell whose center is at a perpendicular distance d from the line charge, for $d = R/2$, and $d = 3R/2$.

Solution:

(a) Because of the symmetry of the charge distribution, the electric field is in the radial direction and depends only on the perpendicular distance from the axis where the line charge is. using Gauss's law $\oint_S \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dA}} = Q_{enc}/\epsilon_0$, where S is the surface of a cylinder of radius r and height L whose axis coincides with the line charge, we find

$$E(r)(2\pi r L) = (\lambda\, L)/\epsilon_0 \quad \rightarrow \quad E_r(r) = \frac{\lambda}{2\pi\epsilon_0\, r}, \qquad 0 < r < 4R$$

and,

$$E(r)(2\pi r L) = (\lambda\, L + \sigma\, 8\pi R L)/\epsilon_0 \quad \rightarrow \quad E_r(r) = \frac{\lambda + 8\pi\sigma R}{2\pi\epsilon_0\, r}, \qquad r > 4R$$

(b) By definition

$$V(3R) - V(R) = \int_{3R}^{R} E_r(r)\, dr = \int_{3R}^{R} \frac{\lambda\, dr}{2\pi\epsilon_0\, r} = \frac{-\lambda \ln 3}{2\pi\epsilon_0}$$

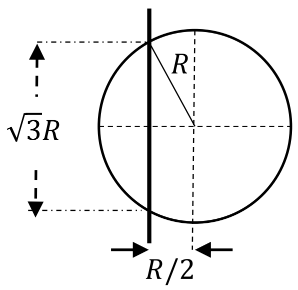
and

$$V(5R) - V(R) = \int_{5R}^{4R} \frac{\lambda + 8\pi\sigma R}{2\pi\epsilon_0\, r}\, dr + \int_{4R}^{R} \frac{\lambda\, dr}{2\pi\epsilon_0\, r} = \frac{\lambda + 8\pi\sigma R}{2\pi\epsilon_0} \ln\frac{4}{5} - \frac{\lambda}{2\pi\epsilon_0} \ln 4 = \frac{-\lambda}{2\pi\epsilon_0} \ln 5 - \frac{4\sigma R}{\epsilon_0} \ln\frac{5}{4}$$

(c) Note that if $d = 3R/2$, we have $Q_{enc} = 0$, therefore, by Gauss's law, $\oint \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dA}} = 0$.

For $d = R/2$, the charge enclosed by the sphere is found by using the following figure.

In this case the charge enclosed by the sphere is $Q_{enc} = \sqrt{3}R\lambda$, therefore, by gauss's law



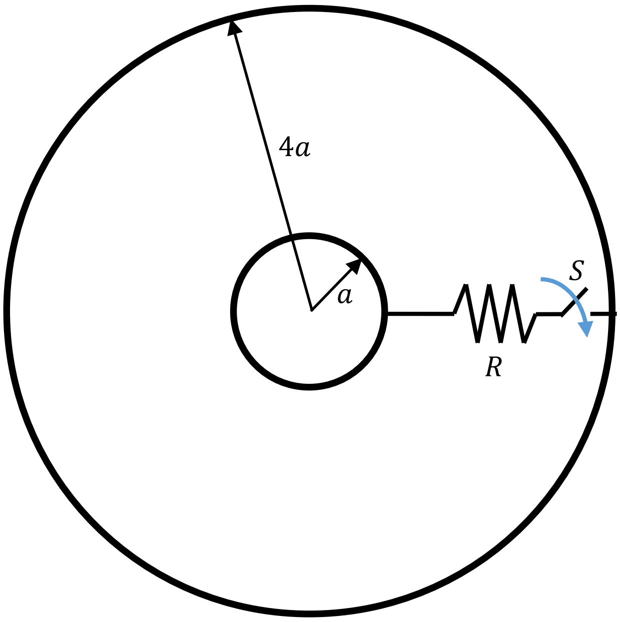
$$\oint \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dA}} = \frac{\sqrt{3}R\lambda}{\epsilon_0} = \frac{2\sqrt{3}d\lambda}{\epsilon_0}$$

3. Two concentric conducting spherical shells of radius a and of radius $4a$ form a spherical capacitor. Initially the net charge on the inner shell is Q while the outer shell is uncharged. At time $t = 0$ the switch S connects the inner shell to the outer shell over a resistor of resistance R. Assume that the resistor is small enough so that it does not change the capacitance of the system.

(a) (5 Pts.) What is the final net charge on the outer shell?

(b) (5 Pts.) What is the capacitance of the system?

(c) (15 Pts.) At what time would the charges on two shells be equal?



Solution:

(a) When the switch S is closed both spheres will be parts of the same conductor. Therefore, all the charge which initially is on the inner sphere will flow to the outer sphere. So the net final charge on the outer shell $Q_{\text{out}} = Q$.

(b) If the inner sphere has charge Q_{in}, the magnitude of the electric field for $a < r$ is found by using Gauss's law as

$$E = \frac{Q_{\text{in}}}{4\pi\epsilon_0 r^2}.$$

Therefore, the potential difference between the two spherical shells is

$$V_{\text{in}} - V_{\text{out}} = \Delta V = \int_a^{4a} \frac{Q_{\text{in}}}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{4a}\right) = \frac{3Q_{\text{in}}}{16\pi\epsilon_0 a}.$$

Since, by definition, $C = Q_{\text{in}}/\Delta V$, capacitance of the spherical capacitor is found as

$$C = \frac{16}{3}\pi\epsilon_0 a\,.$$

(c) When the switch S is closed charge will start to flow from the inner sphere to the outer sphere through the resistor. So, by Ohm's law

$$i(t) = \frac{\Delta V}{R} = \frac{3Q_{\text{in}}}{16\pi\epsilon_0 aR}.$$

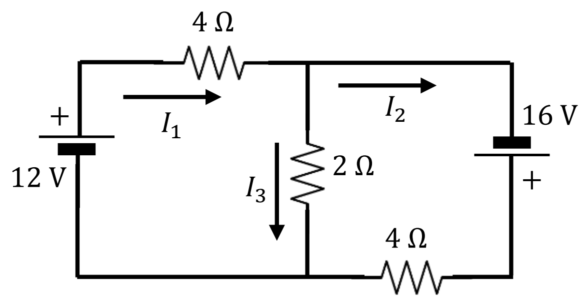
But also $i(t) = -\frac{dQ_{\text{in}}}{dt}$. Therefore,

$$-\frac{dQ_{\text{in}}}{dt} = \frac{3Q_{\text{in}}}{16\pi\epsilon_0 aR} \quad \rightarrow \quad \frac{dQ_{\text{in}}}{Q_{\text{in}}} = -\frac{3\,dt}{16\pi\epsilon_0 aR} \quad \rightarrow \quad Q_{\text{in}}(t) = Qe^{-t/\tau}\,,$$

where $Q = Q_{\text{in}}(t = 0)$ and $\tau = \frac{16}{3}\pi\epsilon_0 aR$ is the time constant. When $Q_{\text{in}} = Q/2$, charges on two spheres will be equal. Therefore, charges on two shells be equal at time

$$\frac{Q}{2} = Qe^{-t/\tau} \quad \rightarrow \quad t = \tau \ln 2 = \frac{16}{3}\pi\epsilon_0 aR \ln 2\,.$$

4. Consider the following circuit.



(a) (9 Pts.) Find the currents I_1, I_2, and I_3 with directions indicated.

(b) (8 Pts.) Find the total power supplied by the batteries.

(c) (8 Pts.) Find the total power dissipated by the resistors.

Solution:

(a) Kirchhoff's junction rule: $I_1 = I_2 + I_3$

Kirchhoff's loop rule 1: $12 - 4I_1 - 2I_3 = 0$, Kirchhoff's loop rule 2: $16 - 4I_2 + 2I_3 = 0$.

Using $I_3 = I_1 - I_2$ in equations given in loop rules 1 and 2, we obtain

$$12 - 4I_1 - 2(I_1 - I_2) = 0 \quad \rightarrow \quad 6I_1 - 2I_2 = 12 \quad \rightarrow \quad 3I_1 - I_2 = 6 \,,$$

and

$$16 - 4I_2 + 2(I_1 - I_2) = 0 \quad \rightarrow \quad -2I_1 + 6I_2 = 16 \quad \rightarrow \quad -I_1 + 3I_2 = 8 \,.$$

Solving these equations, we get

$$I_1 = \frac{13}{4}\text{ A} = 3.25\text{ A}\,, \qquad I_2 = \frac{15}{4}\text{ A} = 3.75\text{ A}\,, \qquad I_3 = I_1 - I_2 = -\frac{1}{2}\text{ A} = -0.5\text{ A}\,.$$

(b) Power supplied by a battery with emf $\mathcal{E}$ supplying a current I is $P_B = \mathcal{E}I$. Hence, total power supplied by the two batteries is

$$P_B = (12\text{ V})\left(\frac{13}{4}\text{ A}\right) + (16\text{ V})\left(\frac{15}{4}\text{ A}\right) = 99\text{ W}\,.$$

(c) Power dissipated by a resistor with resistance R carrying a current I is $P_R = RI^2$. Hence, total power dissipated by the three resistors is

$$P_R = (4\,\Omega)\left(\frac{13}{4}\text{ A}\right)^2 + (4\,\Omega)\left(\frac{15}{4}\text{ A}\right)^2 + (2\,\Omega)\left(\frac{1}{2}\text{ A}\right)^2 = 99\text{ W}\,,$$

as expected.