PHYS 102 – General Physics II Final Exam Solutions

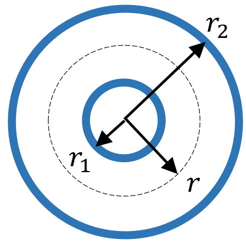
Duration: 120 minutes Friday, 3 January 2020; 18:30

1. A cylindrical capacitor is formed by two concentric cylinders of length ℓ and radii r_1 and r_2, ($r_1 < r_2$), respectively. A material which has dielectric constant κ is used to fill the volume between the plates completely. However, this material is not completely insulating, and it has a resistivity ρ.

(a) (5 Pts.) What is the capacitance of this capacitor?

(b) (5 Pts.) What is the total resistance between the two cylinders?

(c) (10 Pts.) Assume that the total charge in the outer cylinder at time $t = 0$ is Q_0. Calculate the charge $Q(t)$ on the outer cylinder as a function of time.



Solution:

Using Gauss's law where the surface is that of a cylinder of length ℓ and radius r, ($r_1 < r < r_2$) whose cross section is shown in the figure, we find the magnitude of the electric field between two concentric cylinders of infinite length as

$$\oint_S \vec{\boldsymbol{E}} \cdot \overrightarrow{d\boldsymbol{A}} = 2\pi r \ell E(r) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{\lambda}{2\pi\epsilon_0 r}$$

Potential difference between the two cylinders is found as

$$V = \int_{r_1}^{r_2} \vec{\boldsymbol{E}} \cdot \overrightarrow{d\boldsymbol{r}} = \frac{\lambda}{2\pi\epsilon_0} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$$

(a) For a cylindrical capacitor of length ℓ, which is much greater than the separation of the cylinders $r_2 - r_1$, the edge effects can be neglected and we can use the results found above, with $\lambda = Q/\ell$, to write

$$V = \frac{Q}{2\pi\epsilon_0 \ell} \ln \frac{r_2}{r_1} \quad \rightarrow \quad C = \kappa \frac{Q}{V} = \frac{2\pi\kappa\epsilon_0 \ell}{\ln(r_2/r_1)}$$

(b) Since current will flow from one cylinder to the other radially, considering a cylindrical shell of thickness dr and length ℓ concentric with the two cylinders and between them, we have

$$dR = \rho \frac{dr}{2\pi r \ell} \quad \rightarrow \quad R = \int_{r_1}^{r_2} dR = \frac{\rho}{2\pi\ell} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\rho}{2\pi\ell} \ln(r_2/r_1)$$

(c) Charge on the outer cylinder will leak through the medium in between, hence the capacitor will discharge. Since

$$IR = \frac{Q}{C}, \qquad I = -\frac{dQ}{dt} \quad \rightarrow \quad \frac{dQ}{dt} = -\frac{Q}{RC}$$

Integrating, and noting that $RC = \rho\kappa\epsilon_0$, we obtain

$$Q(t) = Q_0 e^{-t/(\rho\kappa\epsilon_0)}$$

2. The position and the velocity of a particle with positive charge q and mass m at time $t = 0$ is given by $\vec{\boldsymbol{r}} = 0$ and $\vec{\boldsymbol{v}} = v_0\ \hat{\boldsymbol{k}}$. The particle is moving in a region where a uniform magnetic field $\vec{\boldsymbol{B}} = B_x\ \hat{\boldsymbol{\imath}} + B_y\ \hat{\boldsymbol{\jmath}}$ exists. (v_0, B_x and B_y are constants)

(a) (10 Pts.) Find the magnitude of the acceleration of the particle in the region.

(b) (5 Pts.) What will be the maximum z-coordinate of the particle during its motion?

(c) (5 Pts.) What is the power supplied by the magnetic field to the particle?

Solution:

(a) The force on the charge in the region at $t = 0$ is

$$\vec{\boldsymbol{F}} = q\ \vec{\boldsymbol{v}_0} \times \vec{\boldsymbol{B}} = q\left(v_0\ \hat{\boldsymbol{k}}\right) \times \left(\ B_x\ \hat{\boldsymbol{\imath}} + B_y\ \hat{\boldsymbol{\jmath}}\right) = \ -qv_0 B_y \hat{\boldsymbol{\imath}} +\ qv_0 B_x\ \hat{\boldsymbol{\jmath}}$$

Using $\vec{\boldsymbol{F}} = m\ \vec{\boldsymbol{a}}$, we find

$$\vec{\boldsymbol{a}} = -\frac{qv_0 B_y}{m}\hat{\boldsymbol{\imath}} + \frac{qv_0 B_x}{m}\ \hat{\boldsymbol{\jmath}} \quad \rightarrow \quad a = \sqrt{\left(\frac{qv_0 B_y}{m}\right)^2 + \left(\frac{qv_0 B_x}{m}\right)^2} = \frac{qv_0}{m}\sqrt{B_x^2 + B_y^2}$$

at time $t = 0$.

Since $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = q\left(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\right) \cdot \vec{\boldsymbol{v}} \equiv 0$, force is always perpendicular to the velocity. Initial velocity has no component parallel to the magnetic field, meaning that the path of the particle is a circle, and the magnitude of the acceleration is constant.

(b) For a uniform circular motion, we have

$$a = \frac{qv_0}{m}\sqrt{B_x^2 + B_y^2} = \frac{v_0^2}{R} \quad \rightarrow \quad R = \frac{m\ v_0}{q\sqrt{B_x^2 + B_y^2}}$$

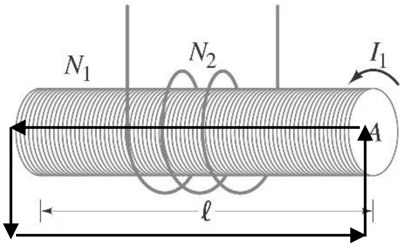
and $z_{max} = R$.

(c) $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = q\left(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\right) \cdot \vec{\boldsymbol{v}} \equiv 0$

3. A long thin solenoid of length ℓ and cross-sectional area A contains N_1 closely packed turns of wire. Wrapped around it is an insulated coil of N_2 turns. Assume all the flux from coil 1 (the solenoid) passes through coil 2.

(a) (10 Pts.) Calculate the mutual inductance.

(b) (10 Pts.) Find the emf induced on coil 2 if the current passing through coil 1 is given by the expression $i_1 = I_1 \cos \omega t$.



Solution:

(a) Using Ampère's law

$$\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{enc} \quad \rightarrow \quad B\ell = \mu_0 N_1 I_1 \quad \rightarrow \quad B = \mu_0 N_1 I_1 / \ell$$

Magnetic flux through the long thin solenoid is

$$\Phi_1 = BA = \mu_0 N_1 I_1 A / \ell$$

Magnetic flux through the second coil is

$$\Phi_2 = N_2 \Phi_1 = \mu_0 N_1 N_2 I_1 A / \ell$$

Therefore

$$M = \frac{\Phi_2}{I_1} = \mu_0 N_1 N_2 A / \ell$$

(b)

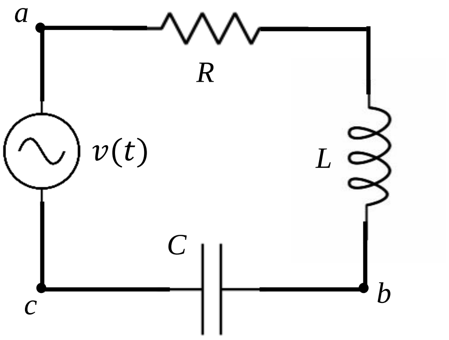
$$\mathcal{E} = -\frac{d\Phi_2}{dt} = (\mu_0 N_1 N_2 A/\ell) \frac{di}{dt} = (\mu_0 N_1 N_2 A/\ell) \omega \, I_1 \sin \omega t$$

4. For the circuit shown in the figure $v(t) = V\cos\omega t$.

(a) (7 Pts.) Find the maximum current I through the circuit.

(b) (6 Pts.) What is the total impedance of the circuit?

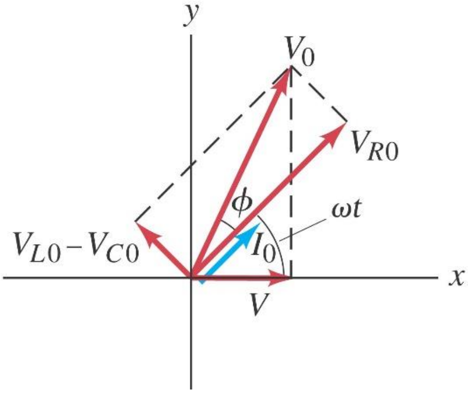
(c) (7 Pts.) At which frequency the maximum voltage across the points a and b is equal to the maximum voltage across the points b and c, i.e., $V_{ab} = V_{bc}$?



Solution:

Because the circuit elements are connected in series, the curent at any instant is the same at every point in the circuit. Thus a single phasor I, with length proportional to the current amplitude, represents the current in all circuit elements. As for the potential differences across the circuit elements, we have $V_R = IR$, in phase with the current, $V_L = IX_L$, leading the current by $\pi/2$, and $V_C = IX_C$, lagging the current by $\pi/2$. Hence, we have the following diagram.

(a)



$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(IR)^2 + (IX_L - IX_C)^2}$$

$$V = I\sqrt{R^2 + (X_L - X_C)^2} = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

(b)

$$I = \frac{V}{Z} \quad \rightarrow \quad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

(c)

$$V_{ab} = I\sqrt{R^2 + (X_L)^2} = I\sqrt{R^2 + (\omega L)^2}$$

$$V_{bc} = \frac{I}{\omega C}$$

$$V_{ab} = V_{bc} \quad \rightarrow \quad R^2 + (\omega L)^2 = \frac{1}{\omega^2 C^2} \quad \rightarrow \quad L^2C^2\,\omega^4 + R^2C^2\,\omega^2 - 1 = 0$$

$$\omega = \frac{1}{\sqrt{2}}\left[\sqrt{\left(\frac{R}{L}\right)^4 + \frac{4}{L^2C^2}} - \left(\frac{R}{L}\right)^2\right]^{1/2}$$

5. Electric fields for two different electromagnetic waves are given as $\overrightarrow{E_1} = \left(300\frac{N}{C}\right)\, \hat{\boldsymbol{\imath}} \cos(k_1 z - 100\, t)$ and $\overrightarrow{E_2} = \left(300\frac{N}{C}\right)\, \hat{\boldsymbol{\jmath}} \cos(k_2 z + 200\, t)$, where z is in meters and t is in seconds. The speed of light is given as $c = 3\ \times 10^8\ m/s.\ (\ k_1 > 0\ ,\ k_2 > 0\ )$

(a) (3 Pts.) What is the ratio of the speed of the first wave and the speed of the second wave?

(b) (3 Pts.) What is the ratio the wavelengths of the two waves $\frac{\lambda_1}{\lambda_2}$?

(c) (3 Pts.) What is the direction of propagation of the first wave?

(d) (3 Pts.) What is the direction of the Poynting vector of the second wave?

(e) (4 Pts.) If both waves are present in vacuum what is the maximum value of magnetic field anywhere at time t=0?

(f) (4 Pts.) If both waves are present in vacuum in which direction is the total energy flux?

Solution:

(a) Both waves are traveling with the same speed. Therefore, the ratio is 1.

(b)

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{\omega_2/c}{\omega_1/c} = \frac{\omega_2}{\omega_1} = \frac{200}{100} = 2$$

(c) From the phase of $\cos(k_1 z - \omega_1\, t)$, we see that the direction of propagation of the first wave is $\hat{\boldsymbol{k}}$.

(d) Poynting vector points in the direction of propagation, which for the phase of $\cos(k_2 z + \omega_2\, t)$ is $-\hat{\boldsymbol{k}}$.

(e) At time $t = 0$ both magnetic fields are maximum at $z = 0$, and for both waves we have

$$B_{max} = E_{max}/c = \frac{300\ (\mathrm{N/C})}{3 \times 10^8\ (\mathrm{m/s})} = 10^{-6}\ \text{Tesla}$$

Since the magnetic component of the two waves are perpendicular to each other,

$$B_{max} = \sqrt{B_{1\,max}^2 + B_{2\,max}^2} = \sqrt{2} \times 10^{-6}\ \text{Tesla}$$

(f) Waves with the same amplitude are traveling in opposite directions. Therefore, the total energy flux is zero.