PHYS 102 – General Physics I Midterm Exam 1 Solutions

Duration: 90 minutes — Saturday, 09 November 2019; 14:00

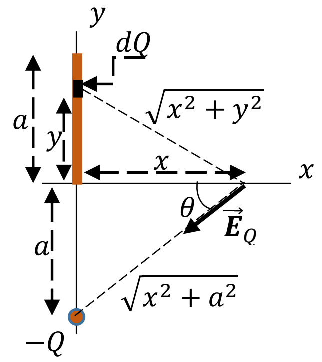
1. Positive charge Q is distributed uniformly along the positive y-axis between $y = 0$ and $y = a$. A negative point charge $-Q$ is fixed on the negative y-axis at $y = -a$.

(a) (8 Pts.) What is the expression for the electric field created by the point charge on the x-axis for $x > 0$?

(b) (9 Pts.) What is the expression for the electric potential created by the line of charge on the x-axis for $x > 0$?

(c) (9 Pts.) What is the x-component of the total electric field on the x-axis for $x > 0$?

(d) (9 Pts.) What is the magnitude of the force exerted on the point charge by the line charge distribution?



Solution:

(a) $|\vec{E}_Q| = \frac{1}{4\pi\epsilon_0}\frac{Q}{(x^2+a^2)}$, $\cos\theta = \frac{x}{\sqrt{x^2+a^2}}$, $\sin\theta = \frac{a}{\sqrt{x^2+a^2}}$. $E_x = \frac{1}{4\pi\epsilon_0}\frac{-Qx}{(x^2+a^2)^{3/2}}$, $E_y = \frac{1}{4\pi\epsilon_0}\frac{-Qa}{(x^2+a^2)^{3/2}}$

$$\vec{E}_Q = \frac{-Q(x\,\hat{\imath} + a\,\hat{\jmath})}{4\pi\epsilon_0(x^2+a^2)^{3/2}}$$

(b) $dV = \frac{dQ}{4\pi\epsilon_0\sqrt{x^2+y^2}}$, $dQ = \frac{Q}{a}dy, \rightarrow V = \int_0^a \frac{Q\,dy}{4\pi\epsilon_0 a\sqrt{x^2+y^2}} = \frac{Q}{4\pi\epsilon_0 a}\int_0^a \frac{dy}{\sqrt{x^2+y^2}}$

$$V = \frac{Q}{4\pi\epsilon_0 a}\left[\ln(a+\sqrt{x^2+a^2}) - \ln x\right] = \frac{Q}{4\pi\epsilon_0 a}\ln\left[\frac{a+\sqrt{x^2+a^2}}{x}\right]$$

(c) For the line charge $E_x = -\frac{dV}{dx} = \frac{Q}{4\pi\epsilon_0 a}\left(\frac{x}{a+\sqrt{x^2+a^2}}\right)\left(\frac{1}{\sqrt{x^2+a^2}} - \frac{a+\sqrt{x^2+a^2}}{x^2}\right) = \frac{Q}{4\pi\epsilon_0 x\sqrt{x^2+a^2}}$.

Therefore, the x-component of the total electric field on the x-axis is

$$E_x = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{x\sqrt{x^2+a^2}} - \frac{x}{(x^2+a^2)^{3/2}}\right]$$

(d) Magnitude of the electric field created by the point charge at the position of the selected infinitesimal section of the line charge is

$$E = \frac{Q}{4\pi\epsilon_0(a+y)^2}$$

The force on the infinitesimal section is therfore

$$dF = dQ\frac{Q}{4\pi\epsilon_0(a+y)^2} = \frac{Q^2dy}{4\pi\epsilon_0 a(a+y)^2} \rightarrow F = \frac{Q^2}{4\pi\epsilon_0 a}\int_0^a \frac{dy}{(a+y)^2} = \frac{Q^2}{4\pi\epsilon_0 a}\left[\frac{-1}{2a} + \frac{1}{a}\right]$$

$$F = \frac{Q^2}{8\pi\epsilon_0 a^2}$$

2. A solid conducting sphere with radius R carries a positive total charge Q. The sphere is surrounded by an insulating shell with inner radius R and outer radius $2R$. The insulating shell has a uniform volume charge density ρ.

(a) (10 Pts.) Find the value of ρ so that the net charge of the entire system is zero.

(b) (20 Pts.) If ρ has the value found in part (a), find the electric field $\vec{\boldsymbol{E}}$ (magnitude and direction) in each of the regions $0 < r < R$, $R < r < 2R$, and $r > 2R$.

Solution:

(a) The volume of the insulating shell is $V = \frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3 = \frac{28}{3}\pi R^3$. Zero net charge requires that

$$-Q = \rho\left(\frac{28}{3}\pi R^3\right) \rightarrow \rho = -\frac{3Q}{28\pi R^3}$$

(b) For $r < R$, $E = 0$ since this region is within the conducting sphere. For $r > 2R$, $E = 0$, since the net charge enclosed by a Gaussian sphere with this radius is zero. For $R < r < 2R$, Gauss's law

$$\oint_S \vec{\boldsymbol{E}} \cdot \overrightarrow{\boldsymbol{dA}} = \frac{Q_{\boldsymbol{enc}}}{\epsilon_0}$$

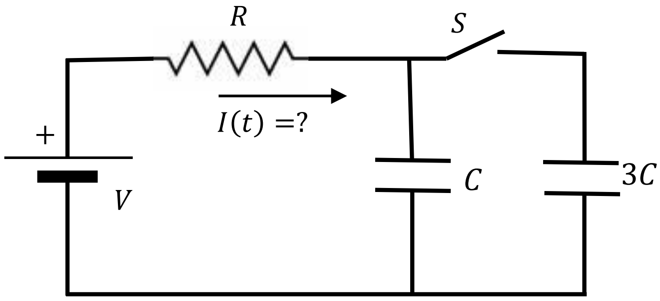
gives

$$E(4\pi r^2) = \frac{Q}{\epsilon_0} + \frac{4\pi\rho}{3\epsilon_0}(r^3 - R^3) \rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2} + \frac{\rho}{3\epsilon_0 r^2}(r^3 - R^3)$$

Using the expression found for ρ in part (a), we find

$$E = \frac{2Q}{7\pi\epsilon_0 r^2} - \frac{Qr}{28\pi\epsilon_0 R^3}$$

3. The switch S in the circuit shown has been open for a very long time and the unconnected capacitor which has capacitance $3C$ is not charged.



(a) (5 Pts.) What is the charge on the other capacitor?

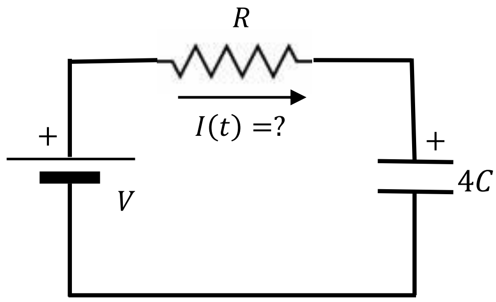
At time $t = 0$ the switch S is closed.

(b) (15 Pts.) Find the current on the resistor I(t) as a function of time after the switch is closed.

(c) (15 Pts.) What is the total energy lost by the battery from the time the switch is closed to infinity?

Solution:

(a) $Q_0 = CV$



(b) After the switch S is closed, we have the following circuit with the $4C$ capacitor initially charged with charge $Q_0 = CV$. Writing the loop rule, we have

$$V - IR - \frac{Q}{4C} = 0$$

Using $I = \frac{dQ}{dt}$, and rearranging the equation, we get

$$\frac{dQ}{4CV - Q} = \frac{dt}{4RC}$$

Integrating both sides noting that $Q(0) = CV$, the result is found as

$$Q(t) = CV\left(4 - 3e^{-t/(4RC)}\right)$$

Since $I = \frac{dQ}{dt}$, we find

$$I(t) = \frac{3V}{4R} e^{-t/(4RC)}$$

(c)

$$\lim_{t\to\infty} Q(t) = 4CV\,, \qquad Q(0) = CV \;\rightarrow\; \Delta Q = 3CV$$

$$\Delta W = V\,\Delta Q \;\rightarrow\; \Delta W = 3CV^2$$