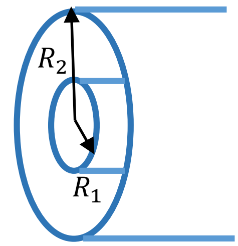
PHYS 102 – General Physics II Final Exam Solutions

Duration: 120 minutes — Wednesday, 22 May 2019

1. Consider a coaxial cable whose inner conductor is a thin hollow tube of radius R_1 and outer conductor is also a thin tube of radius R_2.



(a) (12 Pts.) Determine the capacitance per unit length of the coaxial cable.

(b) (13 Pts.) Determine the inductance per unit length of the coaxial cable.

Solution:

(a) Assume total charge Q in a section of length ℓ. Use Gauss's law assuming a concentric cylindrical Gaussian surface with radius r such that $R_1 < r < R_2$.

$$\Phi_E = (2\pi r\ell)E(r) = \frac{Q_{enc}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \;\rightarrow\; E(r) = \frac{Q}{2\pi\varepsilon_0 r\ell}, R_1 < r < R_2$$

Potential difference between the inner and the outer cylinders is

$$|\Delta V| = \int_{R_1}^{R_2} \frac{Q}{2\pi\varepsilon_0 r\ell} dr = \frac{Q}{2\pi\varepsilon_0\ell}\int_{R_1}^{R_2}\frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0\ell}\ln\left(\frac{R_2}{R_1}\right).$$ Therefore $$C = \frac{Q}{|\Delta V|} = \frac{2\pi\varepsilon_0\ell}{\ln\left(\frac{R_2}{R_1}\right)} \;\rightarrow\; \frac{C}{\ell} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{R_2}{R_1}\right)}.$$

(b) Assume a current I in the inner conductor. Using Ampère's law around a circle of radius r such that $R_1 < r < R_2$,

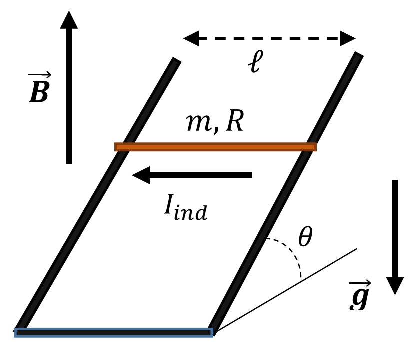
$$\oint_C \vec{\mathbf{B}}\cdot d\vec{\ell} = \mu_0 I \;\rightarrow\; (2\pi r)B(r) = \mu_0 I \;\rightarrow\; B(r) = \frac{\mu_0 I}{2\pi r}.$$

Magnetic flux through a rectangular region of length ℓ between the two concentric cylinders is

$$\Phi_B = \int_{R_1}^{R_2}\frac{\mu_0 I\ell}{2\pi r}dr = \frac{\mu_0 I\ell}{2\pi}\int_{R_1}^{R_2}\frac{dr}{r} = \frac{\mu_0 I\ell}{2\pi}\ln\left(\frac{R_2}{R_1}\right).$$

Hence, inductance per unit length is $\dfrac{\Phi_B}{I\ell} = \dfrac{\mu_0}{2\pi}\ln\left(\dfrac{R_2}{R_1}\right).$

2. A conducting rod of mass *m* and resistance *R* is placed on two parallel conducting rails with no resistance, inclined at an angle θ with the horizontal and connected to each other by another resistanceless rail at the bottom, as shown in the figure. Assume that there is no friction between the rod and the rails. Throughout the region, a uniform magnetic field of magnitude *B* exists in the vertically upward direction. The rod starts from rest and slides down the incline under the action of the constant gravitational force.



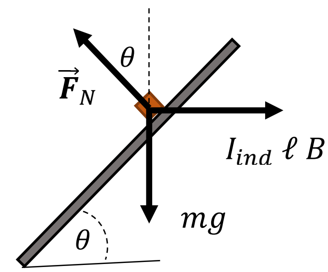
(*a*) (4 Pts) What will be the direction of the induced current? (Show on the figure.)

(*b*) (7 Pts) what will be its terminal speed (i.e. $t \to \infty$)?

(*c*) (7 Pts.) What is the instantaneous power dissipated by the resistance *R* at terminal speed?

(*d*) (7 Pts.) What is the speed of the rod as a function of time?

Solution:



(*b*) Magnetic flux through the rectangle is $\Phi_B = BA\cos\theta = x(t)\ell B\cos\theta$, where *x* is the side of the rectangle. As the bar slides down flux decreases inducing a current in the loop.

$$|\mathcal{E}| = \frac{d\Phi_B}{dt} = \frac{dx}{dt}\ell B\cos\theta = v(t)\ell B\cos\theta \;\; \to \;\; I_{ind} = \frac{|\mathcal{E}|}{R} = \frac{v(t)\ell B\cos\theta}{R}.$$

The magnetic force on the rod is $F_M = I_{ind}\ell B \;\; \to \;\; F_M = \dfrac{v(t)\ell^2 B^2\cos\theta}{R}$. Writing Newton's second law for the rod,

$mg\sin\theta - F_M\cos\theta = ma$, we see that at terminal speed $a = 0$ and $mg\sin\theta - \dfrac{v_T\ell^2B^2\cos^2\theta}{R} = 0$, hence

$$v_T = \frac{Rmg\sin\theta}{\ell^2B^2\cos^2\theta}.$$

(*c*) $P = I_{ind}^2 R = \dfrac{Rm^2g^2}{\ell^2B^2}\tan^2\theta$.

(*d*) For $t > 0$, we have

$$mg\sin\theta - \frac{v(t)\ell^2B^2\cos^2\theta}{R} = m\frac{dv(t)}{dt}$$

whose solution is

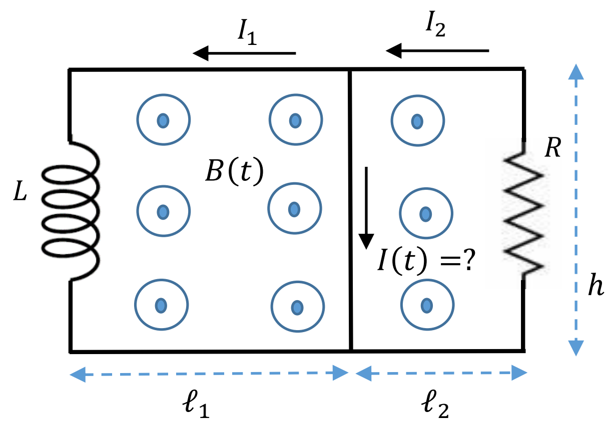
$$v(t) = v_T\left(1 - e^{-t/\tau}\right), \text{ where } \tau = \frac{Rm}{B^2\ell^2\cos^2\theta}.$$

3. An inductor of inductance L is connected to a resistor R with ideal wires as shown in the figure to create a circuit. The circuit has two rectangular meshes of dimensions (ℓ_1, h) and (ℓ_2, h). The circuit is under a homogenous but time dependent magnetic field $B(t) = B_0 cos(\omega t)$.

(a) (9 Pts.) Find the current $I(t)$ over the wire dividing the circuit to two parts as shown in the figure. Assume that the magnetic field has been oscillating for a long time so that AC steady state is reached.

(b) (8 Pts.) What is the maximum value of $I(t)$?

(c) ((8 Pts.) What is the average power dissipated on the resistor?



Solution:

(a) $I(t) = I_2 - I_1$ where

$$RI_2 = -\frac{dB}{dt} h\ell_2 \;\rightarrow\; I_2 = \frac{B_0 h \ell_2 \omega}{R}\sin(\omega t), \text{ and}$$

$$L\frac{dI_1}{dt} = -\frac{dB}{dt} h\ell_1 \;\rightarrow\; I_1 = -\frac{B_0 h \ell_1}{L}\cos(\omega t).$$

Therefore, we have

$$I(t) = B_0 h\left[\frac{\ell_2 \omega}{R}\sin(\omega t) + \frac{\ell_1}{L}\cos(\omega t)\right].$$

(b) One can write the result as

$$I(t) = I_0 \cos(\omega t - \phi), \text{ where } I_{\max} = I_0 = B_0 h\sqrt{\frac{\ell_1^2}{L^2} + \frac{\ell_2^2 \omega^2}{R^2}}, \text{ and } \tan\phi = \frac{\omega \ell_2 L}{\ell_1 R}.$$

(c) Average power dissipated on the resistor is

$$P_{av} = I_{2rms}^2 R = \frac{1}{2} I_{2\max}^2 R = \frac{B_0^2 h^2 \ell_2^2 \omega^2}{2R}.$$

4. An electromagnetic wave has the following electric field

$$\vec{E}(x,y,z) = 200\left(\frac{N}{C}\right)\cos\big(400\pi\, t - k(y+z)\big)\,\hat{\boldsymbol{\imath}}$$

Where (x, y, z) are given in meters and t is given in seconds. If the speed of light is $c = 3\ 10^8 m/s$, answer the following questions about this electromagnetic wave.

(*a*) (3 Pts.) What is the correct SI units for k?

(*b*) (3 Pts.) What is the numerical value of k (use $\pi = 3$)?

(*c*) (3 Pts.) What is the frequency of this wave?

(*d*) (3 Pts.) In which direction is this electromagnetic wave propagating? (Give a vector as your answer).

(*e*) (3 Pts.) What is the wavelength of this wave?

(*f*) (3 Pts.) In which direction is the magnetic field of this wave at the point $x = 0, y = 0, z = 0$ at time $t = 0$?

The Poynting vector of an electromagnetic wave is defined as

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B}$$

(*g*) (3 Pts.) What is the direction of the Poynting vector for this wave?

(*h*) (4 Pts.) What are the correct SI units for Poynting vector in terms of the base SI units (meter, second, kilogram, Ampere)?

Solution:

(*a*) $k = \dfrac{2\pi}{\lambda} \;\rightarrow\; [k] = \text{m}^{-1}$

(*b*) $\omega = 400\pi$ and $k = \dfrac{\omega}{c} = \dfrac{400\pi}{3\times10^{8}} = 4\times10^{-6}\ \text{m}^{-1}$

(*c*) $f = \dfrac{\omega}{2\pi} = 200\,Hz$

(*d*) $\hat{\mathbf{j}}+\hat{\mathbf{k}}$ direction.

(*e*) $\lambda = \dfrac{2\pi}{k} = \dfrac{c}{f} = \dfrac{3\times10^{8}}{200} = 1.5\times10^{6}\ \text{m}$

(*f*) Since the direction of propagation is $\vec{\mathbf{E}}\times\vec{\mathbf{B}}$, $\vec{\mathbf{B}}$ must be in $\hat{\mathbf{j}}-\hat{\mathbf{k}}$ direction.

(*g*) $\hat{\mathbf{j}}+\hat{\mathbf{k}}$

(*h*) $\oint \vec{\mathbf{B}}\, d\vec{\ell} = \mu_0 I \;\rightarrow\; \left[\dfrac{B}{\mu_0}\right] = \dfrac{\text{A}}{\text{m}} = \dfrac{\text{C}}{\text{m s}}$, and $[E] = \dfrac{N}{C} = \dfrac{\text{kg m}}{\text{C s}^2}$. Therefore, $\left[\dfrac{E\,B}{\mu_0}\right] = \dfrac{\text{W}}{\text{m}^2} = \dfrac{\text{kg}}{\text{s}^3}$.