PHYS 102 – General Physics II Midterm Exam 2 Solution

Duration: 120 minutes — Saturday, 13 April 2019, 10:00

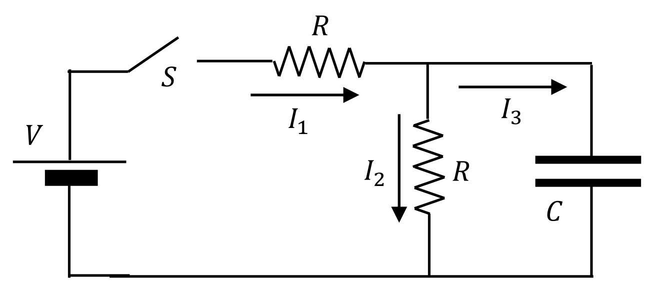
1. The capacitor in the circuit shown is initially uncharged and the switch S is closed at time $t = 0$.

(a) (5 Pts.) Find the currents I_1, I_2 and I_3 at $t = 0$.

(b) (5 Pts.) Find the currents I_1, I_2 and I_3 as $t \to \infty$.

(c) (5 Pts.) What is the charge on the capacitor as $t \to \infty$?

(d) (10 Pts.) Find the charge on the capacitor for $t > 0$.



Solution:

(a) At $t = 0$ the uncharged capacitor behaves like a short circuit. Therefore, $I_2 = 0$, and $I_1 = I_3 = V/R$.

(b) In the limit $t \to \infty$ the capacitor becomes charged, and we have $I_3 = 0$, and $I_1 = I_2 = V/(2R)$.

(c) In the limit $t \to \infty$ the potential difference on the capacitor is $V_C = I_2 R = V/2$. Therefore, charge on the capacitor is $Q(t \to \infty) = CV_C = CV/2$.

(d) For $t > 0$, writing Kirchhoff's lop and junction rules, we have $V - RI_1 - RI_2 = 0$, $V_C - RI_2 = 0$, $I_1 = I_2 + I_3$.

Furthermore, $I_3 = \frac{dQ}{dt}$ and $V_C = \frac{Q}{C}$. Therefore, $I_2 = \frac{V_C}{R} = \frac{Q}{RC}$, and

$$V - 2RI_2 - RI_3 = 0 \quad \to \quad \frac{V}{R} - \frac{2Q}{RC} = \frac{dQ}{dt}$$

Integrating this equation with the fact that $Q(0) = 0$, we find

$$Q(t) = \frac{CV}{2}\left(1 - e^{-\frac{2t}{RC}}\right)$$

Check that $Q(t \to \infty) = CV/2$.

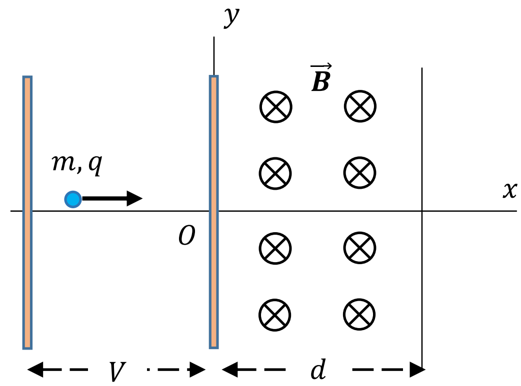
2. A particle with charge q ($q > 0$) and mass m which is initially at rest accelerates through a potential difference V and enters into a region $0 < x < d$, where there is a uniform magnetic field of magnitude B with direction perpendicular to the plane of the paper and inward. Use the coordinate system shown in the figure to answer the following questions in terms of m, q, V, B and d. (Gravitational force on the particle is negligible.)

(a) (5 Pts.) What will be the velocity of the particle as it enters the magnetic field at $x = 0$?

(b) (5 Pts.) In which direction ($+\hat{\boldsymbol{j}}$ or $-\hat{\boldsymbol{j}}$) will the particle be deflected in the region $x > 0$, and what will be its trajectory? Explain why.

(c) (10 Pts.) What is the condition on d if the particle is to exit the magnetic field to enter the region $x > d$?

(d) (5 Pts.) Assuming that the condition found in part (c) holds, what will be the speed of the particle when it exits the region?



Solution:

(a) $\frac{1}{2}mv^2 = qV \quad \rightarrow \quad v = \sqrt{2qV/m}$

(b) $\vec{\boldsymbol{F}}_m = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$, therefore the particle will be deflected upward ($+\hat{\boldsymbol{j}}$ direction) because at $x = 0$

$$\vec{\boldsymbol{F}}_m = qv\,\hat{\boldsymbol{\imath}} \times B\left(-\hat{\boldsymbol{k}}\right) = qvB\,\hat{\boldsymbol{\jmath}}$$

Its trajectory will be a circle on the xy −plane because the particle has no initial velocity in the z −direction, and the force is always perpendicular to the velocity.

(c) The radius of the circular trjectory is found as $F_m = qvB = \frac{mv^2}{r} \quad \rightarrow \quad r = \frac{mv}{qB}$.

If the particle is to exit the magnetic field to enter the region $x > d$, we need to have $r > d$. Therefore,

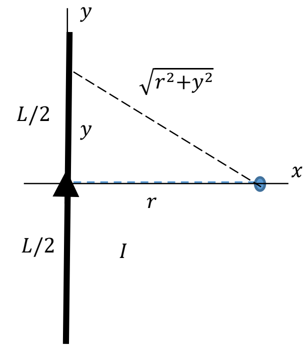
$$d < \frac{mv}{qB} \quad \rightarrow \quad d < \frac{m}{qB}\sqrt{2qV/m} \quad \rightarrow \quad d < \sqrt{\frac{2mV}{qB^2}}$$

(d) The speed of the particle will not change during its uniform circular motion. Therefore, when it exits the region

$$v = \sqrt{2qV/m}$$

3. A straight wire segment of length L is carrying a current I.

(*a*) (13 Pts.) Find the magnitude of the magnetic field at the point which is a perpendicular distance r away from the midpoint of the wire.

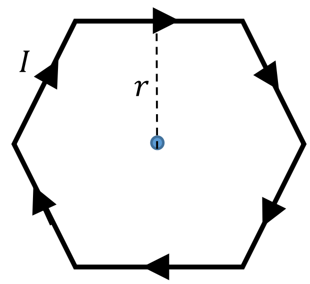


Using your result in (*a*):

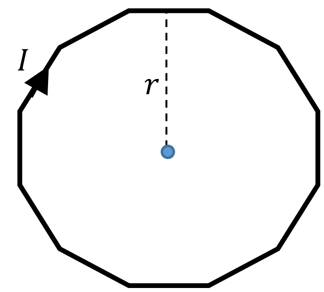
(*b*) (6 Pts.) Find the magnetic field at the center of a regular hexagon with inscribed circle radius r, carrying a current I.

(*c*) (6 Pts.) Find the magnitude of the magnetic field at the center of a regular N sided polygon with inscribed circle radius r, carrying a current I for $N \to \infty$.

Solution: Use Biot – Savart law $\overrightarrow{\boldsymbol{dB}} = \frac{\mu_0 I}{4\pi}\frac{\overrightarrow{\boldsymbol{d\ell}}\times\hat{\boldsymbol{r}}}{r^2}$



Hexagon: 6



N-gon: N sides

Choosing the current in the y −direction and the point on the perpendicular bisector at $(r, 0)$ on the x −axis, we have

$$\overrightarrow{\boldsymbol{d\ell}} = dy\,\hat{\boldsymbol{j}}\,, \qquad r = \sqrt{r^2+y^2}\,, \hat{\boldsymbol{r}} = \frac{r\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}}}{\sqrt{r^2+y^2}}$$

$$\to \quad \overrightarrow{\boldsymbol{d\ell}} \times \hat{\boldsymbol{r}} = \frac{r\,dy\,(-\hat{\boldsymbol{k}})}{\sqrt{r^2+y^2}}$$

Hence

$$\vec{\boldsymbol{B}} = \frac{\mu_0 I}{4\pi}\int_{-L/2}^{L/2}\frac{r\,dy\,(-\hat{\boldsymbol{k}})}{(r^2+y^2)^{3/2}} \quad \to \quad \frac{\mu_0 I r(-\hat{\boldsymbol{k}})}{4\pi}\int_{-L/2}^{L/2}\frac{dy}{(r^2+y^2)^{3/2}}$$

The integral can easily be evaluated using $y = r\tan\theta$ substitution, and the result becomes

$$\vec{\boldsymbol{B}} = \frac{\mu_0 I L}{2\pi r\sqrt{L^2+4r^2}}(-\hat{\boldsymbol{k}})$$

(*b*) For a hexagon $\frac{L}{2r} = \tan\pi/6 = \frac{1}{\sqrt{3}} \quad \to \quad L = 2r/\sqrt{3}$, and there are six segments which contribute to the magnetic field in the same direction. Therefore,

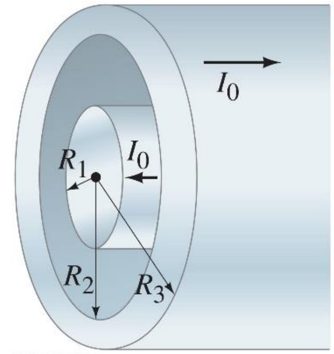
$$\vec{\boldsymbol{B}} = \frac{3\mu_0 I}{2\pi r}(-\hat{\boldsymbol{k}})$$

(*c*) For N-gon $L = 2r\tan\pi/N$ and there are N segments. As $N \to \infty$ the N-gon will become a circle. The magnitude of the magnetic field at the center of the circle will be

$$B = \lim_{N\to\infty}\frac{N\mu_0 I L}{2\pi r\sqrt{L^2+4r^2}} = \frac{\mu_0 I}{2\pi r}\lim_{N\to\infty}\left(N\sin\frac{\pi}{N}\right) = \frac{\mu_0 I}{2r}$$

4. (25 Pts.) A coaxial cable consists of a solid inner conductor of radius R_1, surrounded by a concentric cylindrical tube of inner radius R_2 and outer radius R_3. The conductors carry equal and opposite currents I_0 distributed uniformly across their cross sections. Determine the magnetic field at a distance r from the axis for:

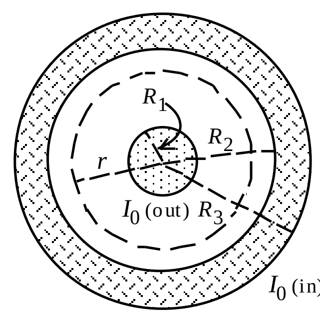
(*a*) $0 \le r < R_1$; (*b*) $R_1 < r < R_2$; (*c*) $R_2 < r < R_3$; (*d*) $r > R_3$.



Solution:

Because of the cylindrical symmetry, the magnetic fields will be circular. In each case, we can determine the magnetic field using Ampere's law with concentric loops. The current densities in the wires are given by the total current divided by the cross-sectional area.

$$J_{\text{inner}} = \frac{I_0}{\pi R_1^2} \qquad J_{\text{outer}} = -\frac{I_0}{\pi\left(R_3^2 - R_2^2\right)}$$



(*a*) Inside the inner wire the enclosed current is determined by the current density of the inner wire.

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{\text{encl}} = \mu_0\left(J_{\text{inner}} \pi r^2\right)$$

$$B(2\pi r) = \mu_0 \frac{I_0 \pi r^2}{\pi R_1^2} \rightarrow \boxed{B = \frac{\mu_0 I_0 r}{2\pi R_1^2}}$$

(*b*) Between the wires the current enclosed is the current on the inner wire.

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{\text{encl}} \quad \rightarrow \quad B(2\pi r) = \mu_0 I_0 \quad \rightarrow \quad \boxed{B = \frac{\mu_0 I_0}{2\pi r}}$$

(*c*) Inside the outer wire the current enclosed is the current from the inner wire and a portion of the current from the outer wire.

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{\text{encl}} = \mu_0\left[I_0 + J_{\text{outer}} \pi\left(r^2 - R_2^2\right)\right]$$

$$B(2\pi r) = \mu_0\left[I_0 - I_0 \frac{\pi\left(r^2 - R_2^2\right)}{\pi\left(R_3^2 - R_2^2\right)}\right] \rightarrow \boxed{B = \frac{\mu_0 I_0}{2\pi r} \frac{\left(R_3^2 - r^2\right)}{\left(R_3^2 - R_2^2\right)}}$$

(*d*) Outside the outer wire the net current enclosed is zero.

$$\oint \vec{\mathbf{B}} \cdot d\vec{s} = \mu_0 I_{\text{encl}} = 0 \quad \rightarrow \quad B(2\pi r) = 0 \quad \rightarrow \quad \boxed{B = 0}$$