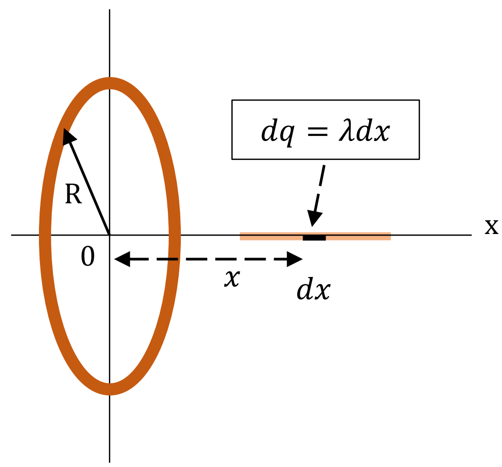
PHYS 102 – General Physics II Midterm Exam 1 Solutions

Duration: 120 minutes Saturday, 9 March 2019, 10:00

1. Consider a thin ring-shaped charge distribution with radius R and total charge Q. If this ring is placed such that its center is at the origin and its symmetry axis is along the x-direction, the electric field it creates along the x-axis is

$$\vec{\boldsymbol{E}}(x) = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+R^2)^{3/2}}\hat{\boldsymbol{\imath}}$$



(*a*) (5 Pts.) What will be the electrical force on a charge -q placed at $x = d$ in the above field?

(*b*) (15 Pts.) If, instead of a point charge, we place a thin rod with total charge $-q$ distributed uniformly over its length ℓ lying on the x-axis at $d \le x \le d+\ell$, what will be the electrical force on the rod?

(*c*) (5 Pts.) What will be the expression for the electric field on the x-axis if we place another identical ring with its center at $x = a$ and identical symmetry axis?

Solution: (*a*)

$$\vec{\boldsymbol{F}} = -q\vec{\boldsymbol{E}}(x) = \frac{1}{4\pi\epsilon_0}\frac{-qQx}{(x^2+R^2)^{3/2}}\hat{\boldsymbol{\imath}}$$

(*b*) Consider an infinitesimal piece of the rod situated at the point $x, (d \le x \ge d+\ell)$. Infinitesimal amount of charge on the piece will be $dq = \lambda\, dx = -\left(\frac{q}{\ell}\right) dx$. Infinitesimal force on this piece is

$$d\vec{\boldsymbol{F}} = dq\vec{\boldsymbol{E}}(x) = \frac{-qQ}{4\pi\epsilon_0\ell}\frac{x\,dx}{(x^2+R^2)^{3/2}}\hat{\boldsymbol{\imath}}$$

Integrating this expression, we get

$$\vec{\boldsymbol{F}} = \frac{-qQ}{4\pi\epsilon_0\ell}\left(\int_d^{d+\ell}\frac{x\,dx}{(x^2+R^2)^{3/2}}\right)\hat{\boldsymbol{\imath}} = \frac{-qQ}{4\pi\epsilon_0\ell}\left(\frac{1}{\sqrt{d^2+R^2}} - \frac{1}{\sqrt{(d+\ell)^2+R^2}}\right)\hat{\boldsymbol{\imath}}$$

(*c*) Electric field created by the second ring can be obtained by replacing x with $x-a$ in the expression of the electric field created by the first ring. Superposing two results, we find

$$\vec{\boldsymbol{E}}(x) = \frac{Q}{4\pi\epsilon_0}\left(\frac{x}{(x^2+R^2)^{3/2}} + \frac{x-a}{((x-a)^2+R^2)^{3/2}}\right)\hat{\boldsymbol{\imath}}$$

2. Consider a hollow spherical conducting shell with inner radius R_1 and outer radius R_2. We measure and find out that the magnitude of the electric field at the inner surface ($r = R_1$) is E_1, pointing inwards and is uniform on the surface, while the magnitude of the electric field at the outer surface ($r = R_2$) is E_2, pointing outward and is uniform on the surface. Use Gauss's law to answer the following questions in terms of E_1, E_2, R_1, R_2 and ε_0.

(*a*) (6 Pts.) What is the surface charge density σ_1 on the inner surface, and the surface charge density σ_2 on the outer surface of the shell?

(*b*) (6 Pts.) What is the total charge inside the cavity?

(*c*) (6 Pts.) What is the total charge on the shell?

(*d*) (7 Pts.) What is the expression for the electric field in regions $R_1 < r < R_2$ and $R_2 < r < \infty$?

Solution:

(*a*) Presence of a uniform electric field at the inner surface ($r = R_1$) pointing inwards means there must be a spherically-symmetric negative charge distribution inside the hollow spherical shell which induces an equal amount of positive charge distributed uniformly on the inner surface of the shell. Using Gauss's law, we have

$$\int \vec{\boldsymbol{E}} \cdot \overrightarrow{d\boldsymbol{A}} = E_1(4\pi R_1^2) = \frac{Q_{enc}}{\varepsilon_0} = \frac{(4\pi R_1^2)\sigma_1}{\varepsilon_0}$$

which gives $\sigma_1 = \varepsilon_0 E_1$. Similarly $\sigma_2 = \varepsilon_0 E_2$.

(*b*) Since the charge inside the cavity is negative, we have $Q_{enc} = -(4\pi R_1^2)\varepsilon_0 E_1$.

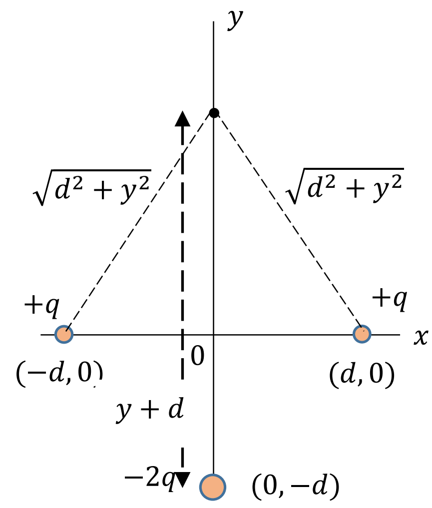
(*c*) A uniform electric field on the outer surface of the shell pointing outward means that the charge dstributed on the outer surface is also positive. Since its density is $\sigma_2 = \varepsilon_0 E_2$, total charge on the outer surface is $Q_{out} = (4\pi R_2^2)\varepsilon_0 E_2$. This means total charge on the spherical shell is

$$Q_{total} = Q_{in} + Q_{out} = (4\pi R_1^2)\varepsilon_0 E_1 + (4\pi R_2^2)\varepsilon_0 E_2 = 4\pi\varepsilon_0(E_1 R_1^2 + E_2 R_2^2)$$

(*d*) Inside the conductor $\vec{\boldsymbol{E}} = 0$, $R_1 < r < R_2$. Due to the spherical symmetry of the charge distribution, electric feld outside the shell is that of a point charge with magnitude equal to the charge on the outer surface of the shell. i.e.,

$$\vec{\boldsymbol{E}} = \frac{Q_{out}}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} = \frac{R_2^2 E_2}{r^2}\hat{\boldsymbol{r}}$$

3. Three point charges are arranged as shown in the figure, with two $+q$ charges fixed at points $(\pm d, 0)$ and a $-2q$ charge fixed at the point $(0, -d)$.



(a) (7 Pts.) Find the expression for the electric potential $V(y)$ for points along the y-axis for $y > 0$.

(b) (6 Pts.) Find the electric field $\vec{\boldsymbol{E}}$ on the y-axis for $y > 0$.

(c) (6 Pts.) Determine the total electrostatic potential energy of the three charges.

(d) (6 Pts.) Find the dipole moment (vector) of the charge configuration.

Solution:

(a)

$$V(y) = \frac{2q}{4\pi\varepsilon_0\sqrt{d^2+y^2}} - \frac{2q}{4\pi\varepsilon_0(y+d)} = \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{\sqrt{d^2+y^2}} - \frac{1}{y+d}\right)$$

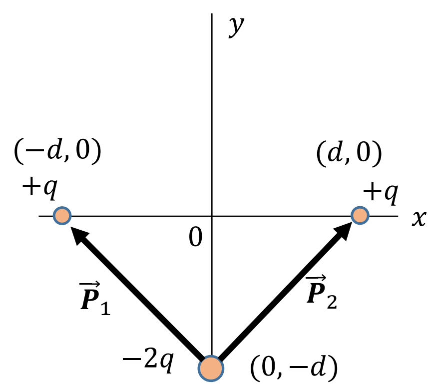
(b)

$$E_y(y) = -\frac{dV}{dy} = \frac{-q}{2\pi\varepsilon_0}\left(\frac{-y}{(d^2+y^2)^{3/2}} + \frac{1}{(y+d)^2}\right)$$

(c)

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{-2q^2}{d\sqrt{2}} + \frac{-2q^2}{d\sqrt{2}} + \frac{q^2}{2d}\right) = \frac{q^2}{4\pi\varepsilon_0 d}\left(\frac{1}{2} - 2\sqrt{2}\right)$$

(d) Given charge distribution is two dipoles as shown, where



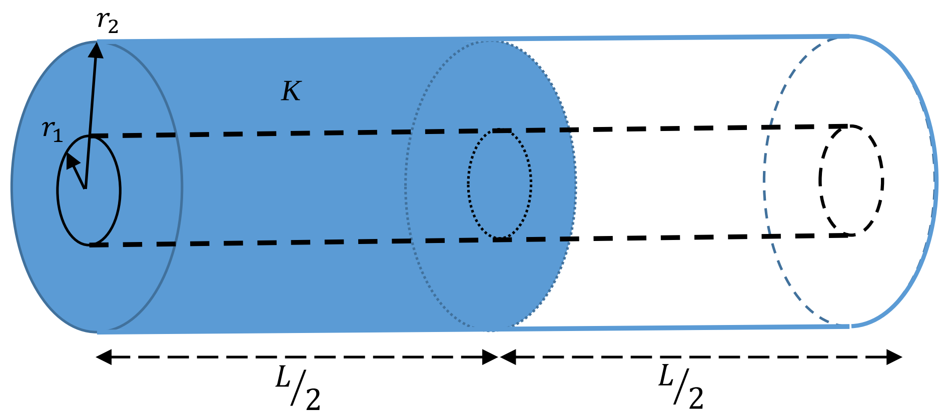
$\vec{\boldsymbol{P}}_1 = -qd\,\hat{\boldsymbol{\imath}} + qd\,\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{P}}_2 = qd\,\hat{\boldsymbol{\imath}} + qd\,\hat{\boldsymbol{\jmath}}$. Therefore

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{P}}_1 + \vec{\boldsymbol{P}}_2 = 2qd\,\hat{\boldsymbol{\jmath}}$$

4. Two coaxial metallic cylindrical pipes of length L are to be used as a capacitor. The radius of the inner pipe is r_1 and the outer pipe is r_2. Half the length of the pipe is filled with material of dielectric constant K while the remaining half is empty. Assume that $L \gg r_1, L \gg r_2$ so that the fringing fields are negligible.

(a) (13 Pts.) Find the total capacitance of the system.

(b) (12 Pts.) If this capacitor is charged with total charge Q what is the amount of charge that is stored in the empty half of the capacitor?



Solution:

(a) The cylinderical capacitor can be considered as two cylindrical capacitors in parallel, one with a dielectric inside. Using Gauss's law, electric field magnitude between two oppositely charged concentric cylinders is found to be

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{Q}{2\pi\varepsilon_0 L r}$$

The potential difference between the iner and the outer cylinder is

$$\Delta V = \int_{r_1}^{r_2} E(r)\,dr = \frac{Q}{2\pi\varepsilon_0 L}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{r_2}{r_1}\right)$$

Capacitance of a cylindrical capacitor is found as

$$\mathrm{C} = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln\left({}^{r_2}/_{r_1}\right)}$$

Equivalent capacitance for parallel combination of capacitors is $C_{eq} = C_1 + C_2$ gives

$$\mathrm{C}_{eq} = \frac{\pi K\varepsilon_0 L}{\ln\left({}^{r_2}/_{r_1}\right)} + \frac{\pi\varepsilon_0 L}{\ln\left({}^{r_2}/_{r_1}\right)} = \frac{\pi\varepsilon_0 L}{\ln\left({}^{r_2}/_{r_1}\right)}(1+K)$$

(b) The total charge Q is divided between two parallel components of the capacitor, so $Q = Q_1 + Q_2$, where Q_2 is the charge that is stored in the empty half of the capacitor. Since the potential difference between the two parts is same,

$$V = \frac{Q_1}{KC} = \frac{Q_2}{C} \quad \Longrightarrow \quad Q_1 - KQ_2 = 0$$

Solving for Q_2, we get

$$Q_2 = \frac{Q}{1+K}$$