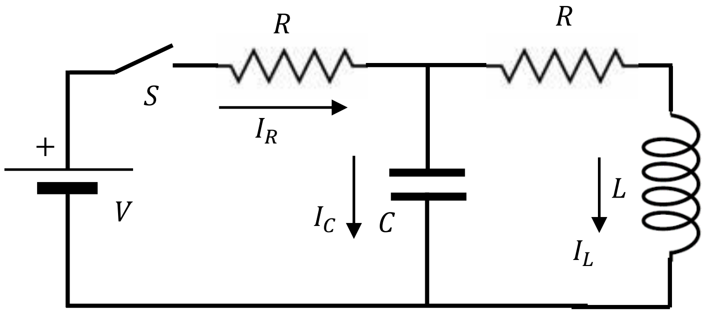
PHYS 102 – General Physics II Final Exam Solutions

Friday, 04 January 2019, 15:30

1. Consider the circuit in the figure with the capacitor initially uncharged and switch S closed at time $t = 0$.

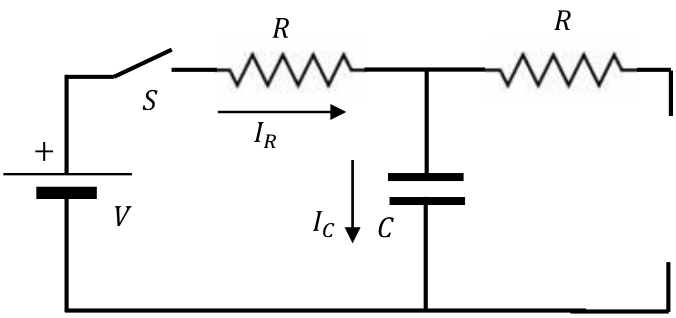
(*a*) (12 Pts.) Find currents I_R, I_L and I_C right after the switch is closed (at time $t = 0$)?

(*b*) (13 Pts.) Find the charge on the capacitor in the limit $t \to \infty$.



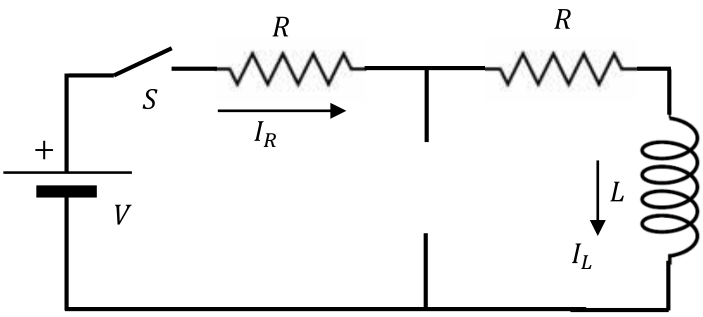
Solution:

(*a*) Since the current through the inductor at time $t = 0$ is zero, the equivalent circuit is:



Therefore, $I_R = I_C = \frac{V}{R}$, $I_L = 0$.

(*b*) In the limit $t \to \infty$, the capacitor becomes fully charged and we have $I_C = 0$. In this case the equivalent circuit is:

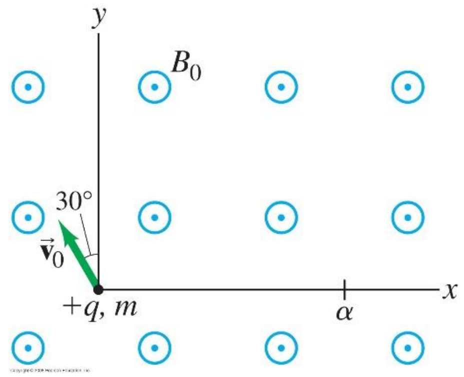


Therefore, $I_R = I_L = \frac{V}{2R}$, $I_C = 0$.

Since $V_C = V - \frac{V}{2} = \frac{V}{2}$, and $Q = CV_C$, final charge on the capacitor is $Q = \frac{1}{2} CV$.

2. A particle with charge $+q$ and mass *m* travels in a uniform magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{k}}$. At time $t = 0$, the particle's velocity is $\vec{\mathbf{v}}_0$, and it lies in the *xy* plane directed at an angle of 30° with respect to the *y* axis as shown. At a later time $t = t_\alpha$ the particle will cross the *x* axis at $x = \alpha$. Determine in terms of *q*, *m*, v_0. and B_0:

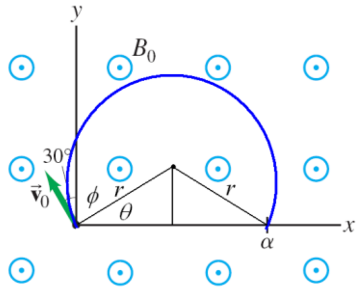
(*a*) (15 Pts.) α ; and (*b*) (10 Pts.) t_α .



Solution:

(*a*) Since the velocity is perpendicular to the magnetic field,

the particle will follow a circular trajectory in the *x*-*y* plane of radius *r*. The radius is found using the centripetal acceleration.



$$qvB = \frac{mv^2}{r} \quad \rightarrow \quad r = \frac{mv}{qB}$$

From the figure we see that the distance α is the chord distance, which is twice the distance $r\cos\theta$ Since the velocity is perpendicular to the radial vector, the initial direction and the angle ϕ are complementary angles. The angles ϕ and θ are also complementary angles, so $\theta = 30°$.

$$\alpha = 2r\cos\theta = \frac{2mv_0}{qB_0}\cos 30° = \sqrt{3}\frac{mv_0}{qB_0}$$

(*b*) From the diagram, we see that the particle travels a circular path, that is 2ϕ short of a complete circle. Since the angles ϕ and θ are complementary angles, so $\phi = 60°$. The trajectory distance is equal to the circumference of the circular path times the fraction of the complete circle. Dividing the distance by the particle speed gives t_α.

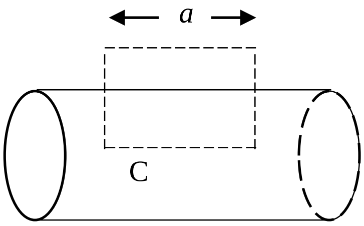
$$t_\alpha = \frac{\ell}{v_0} = \frac{2\pi r}{v_0}\left(\frac{360° - 2(60°)}{360°}\right) = \frac{2\pi}{v_0}\frac{mv_0}{qB_0}\left(\frac{2}{3}\right) = \frac{4\pi m}{3qB_0}.$$

3. A cylinder of length ℓ and radius r is tightly wound by a copper wire making N turns. Assume that the wire is wound tightly ($N \gg 1$) and the cylinder is narrow ($\ell \gg r$) so that the magnetic field outside the solenoid is negligible. If the wire is carrying a current, I:

(a) (8 Pts.) find the magnetic field magnitude inside the solenoid;

(b) (9 Pts.) find the total energy stored by the magnetic field inside the solenoid;

(c) (8 Pts.) find the self-inductance of the solenoid.



Solution:

(a) We apply Ampère's law to the path shown in the figure.

$$\oint_c \vec{\mathbf{B}} \cdot d\vec{\ell} = \int_{c1} \vec{\mathbf{B}} \cdot d\vec{\ell} + \int_{c2} \vec{\mathbf{B}} \cdot d\vec{\ell} + \int_{c3} \vec{\mathbf{B}} \cdot d\vec{\ell} + \int_{c4} \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I_{enc} ,$$

where $c1, c2, c3$ and $c4$ denote the four sides of the rectangle. The magnetic field outside the solenoid is zero, while $\vec{\mathbf{B}} \cdot d\vec{\ell} = 0$ on the two vertical sides of the path C. Therefore, we have

$$\oint_c \vec{\mathbf{B}} \cdot d\vec{\ell} = Ba = \mu_0 I \frac{N}{\ell} a \;\; \rightarrow \;\; B = \mu_0 I \frac{N}{\ell} .$$

(b) Energy density of the magnetic field is $u_B = \frac{1}{2\mu_0} B^2 = \frac{\mu_0 I^2 N^2}{2\ell^2}$. Since $E_B = u_B V$, where V is the volume inside the solenoid, we have

$$E_B = \frac{\mu_0 I^2 N^2}{2\ell^2} \left(\pi r^2 \ell \right) = \frac{1}{2} \frac{\mu_0 \pi r^2 N^2}{\ell} I^2 .$$

(c) The flux of the magnetic field through a cross-section of the solenoid is $\Phi_B = B\left(\pi r^2\right) = \mu_0 \pi r^2 \frac{N}{\ell} I$.

Since for N turns $\Phi_N = N\Phi_B = \mu_0 \pi r^2 \frac{N^2}{\ell} I$, and $\Phi_N = LI$, we have

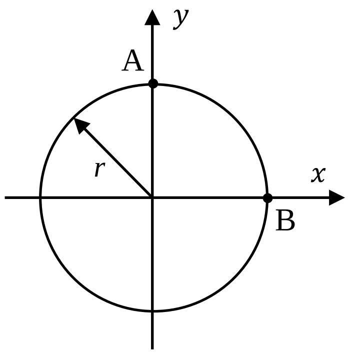
$$LI = \mu_0 \pi r^2 \frac{N^2}{\ell} I \;\; \rightarrow \;\; L = \mu_0 \pi r^2 \frac{N^2}{\ell} .$$

We see that $E_B = \frac{1}{2} L I^2$.

4. In a circular region of space on the xy-plane there exists a uniform magnetic field which changes in time according to the expression $\vec{\mathbf{B}} = B_0(1-e^{-t/\tau})\hat{\mathbf{k}}$, where B_0 and τ are constants.

(*a*) (13 Pts.) Find the expression for the magnitude of the electric field at point A of the figure. Draw the electric field vector at point A on the figure.

(*b*) (12 Pts.) Draw the Poynting vector at point B of the figure. What is its magnitude?



Solution:

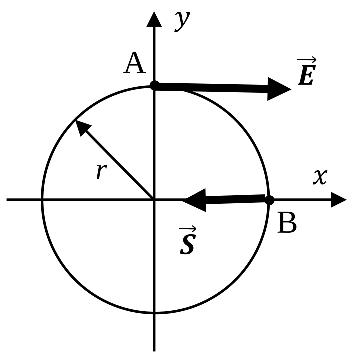
(*a*) Magnetic flux through the circular region enclosed by the circle C of radius r is

$$\Phi_B = B(\pi r^2) = \pi r^2 B_0(1-e^{-t/\tau}).$$

Faraday's law $\mathcal{E}_{ind} = -\dfrac{d\Phi_B}{dt}$.

Since $\mathcal{E}_{ind} = \oint_C \vec{\mathbf{E}}\cdot d\vec{\ell} = E(2\pi r)$, we have $|E(r)|(2\pi r) = \dfrac{1}{\tau}\pi r^2 B_0 e^{-t/\tau} \;\rightarrow\; |E(r)| = \dfrac{B_0}{2\tau} r e^{-t/\tau}$.

Magnetic flux is increasing, therefore, according to Lenz's law, the electric field induced on the circle C is in clockwise direction.



(*b*) $\vec{\mathbf{S}} = \dfrac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}}$. Electric and magnetic fields are perpendicular to each other. Therefore

$$S = \frac{B_0^2 r}{2\mu_0\tau}(1-e^{-t/\tau})e^{-t/\tau}$$, direction being $-\hat{\mathbf{j}}\times\hat{\mathbf{k}} = -\hat{\mathbf{i}}$, i.e., inward at the point B.