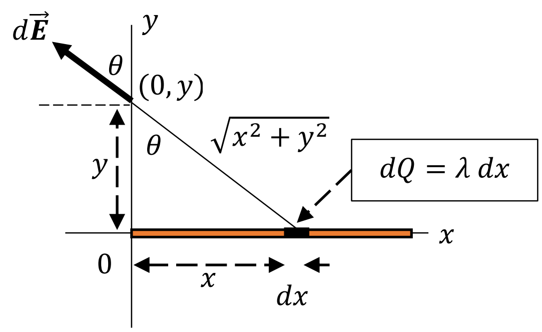
PHYS 102 – General Physics II Midterm Exam 1 Solutions

Duration: 120 minutes | Saturday, 27 October 2018, 14:00

1. (25 Pts.) A line charge with uniform charge density $+\lambda$, placed on the x −axis, extends from $x = 0$ to $x \to \infty$. Determine the electric field (vector) $\vec{E}$ at point $(0, y)$.

Solution:



$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2}, \; r = \sqrt{x^2 + y^2}$$

$$dE_x = -dE\sin\theta, \; dE_y = dE\cos\theta$$

$$\sin\theta = \frac{x}{\sqrt{x^2+y^2}}, \; \cos\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$E_x = \frac{-\lambda}{4\pi\varepsilon_0}\int_0^\infty \frac{x\,dx}{\left(x^2+y^2\right)^{3/2}}$$. Let $u^2 = x^2 + y^2, \; \to \; u\,du = x\,dx$. Integral becomes $\int \frac{x\,dx}{\left(x^2+y^2\right)^{3/2}} = \int \frac{du}{u^2} = -\frac{1}{u}$.

Therefore

$$E_x = \frac{-\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2}}\right]_0^\infty = \frac{-\lambda}{4\pi\varepsilon_0 y}.$$

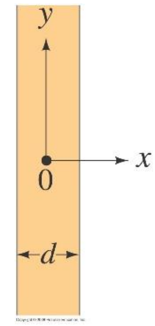
Similarly,

$$E_y = \frac{\lambda\, y}{4\pi\varepsilon_0}\int_0^\infty \frac{dx}{\left(x^2+y^2\right)^{3/2}}$$. Let $x = y\tan\theta, \; \to \; dx = y\sec^2\theta\, d\theta, \; x^2+y^2 = y^2\left(1+\tan^2\theta\right) = y^2\sec^2\theta$

$$\int_0^\infty \frac{dx}{\left(x^2+y^2\right)^{3/2}} = \int_0^{\pi/2} \frac{y\sec^2\theta\, d\theta}{y^3\sec^3\theta} = \frac{1}{y^2}\int_0^{\pi/2}\cos\theta\, d\theta = \frac{1}{y^2}$$. Hence

$$E_y = \frac{\lambda}{4\pi\varepsilon_0 y}.$$

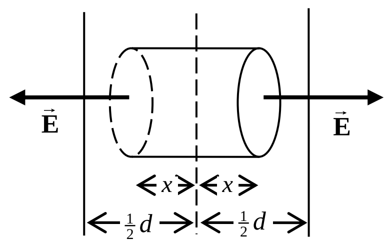
2. (25 Pts.) A flat slab of nonconducting material carries a uniform charge per unit volume, ρ_E. The slab has thickness *d* which is small compared to the height and breadth of the slab. Determine the electric field as a function of *x* inside the slab, and outside the slab (at distances much less than the slab's height or breadth). Take the origin at the center of the slab.



Solution:

Because the slab is very large, and we are considering only distances from the slab much less than its height or breadth, the symmetry of the slab results in the field being perpendicular to the slab, with a constant magnitude for a constant distance from the center. We assume that $\rho_E > 0$ and so the electric field points away from the center of the slab.

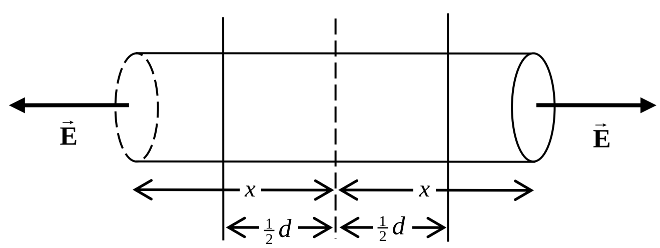
(*a*) To determine the field inside the slab, choose a cylindrical Gaussian surface, of length $2x < d$ and cross-sectional area *A*. Place it so that it is centered in the slab. There will be no flux through the curved wall of the cylinder. The electric field is parallel to the surface area vector on both ends, and is the same magnitude on both ends. Apply Gauss's law to find the electric field at a distance $x < \frac{1}{2}d$ from the center of the slab.



$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_{\text{ends}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \int_{\text{side}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_{\text{ends}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + 0 = \frac{Q_{\text{encl}}}{\varepsilon_0} \rightarrow 2EA = \frac{\rho(2xA)}{\varepsilon_0} \rightarrow$$

$$\boxed{E_{\text{inside}} = \frac{\rho x}{\varepsilon_0};\ |x| < \tfrac{1}{2}d}$$

(*b*) Use a similar arrangement to determine the field outside the slab. Now let $2x > d$.

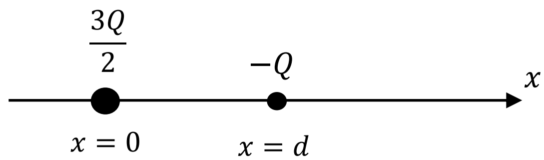


$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \int_{\text{ends}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{encl}}}{\varepsilon_0} \rightarrow$$

$$2EA = \frac{\rho(dA)}{\varepsilon_0} \rightarrow \boxed{E_{\text{outside}} = \frac{\rho d}{2\varepsilon_0};\ |x| > \tfrac{1}{2}d}$$

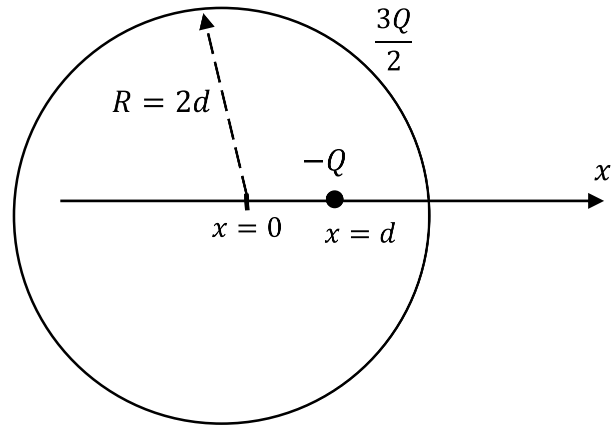
The direction of the electric field is as indicated in the figure. Notice that electric field is continuous at the boundary of the slab.

3. A charge $3Q/2$ is placed at the origin and another charge $-Q$ is placed at $x = d$ on the x axis.



(*a*) (6 Pts.) Find all the points (except infinity) on the x axis at which the electric potential is zero.
(*b*) (6 Pts.) Find all the points (except infinity) on the x axis at which the electric field is zero.

A non-conducting thin spherical shell of radius $R = 2d$ centered at the origin is uniformly charged with total charge $3Q/2$. Another charge $-Q$ is placed inside the spherical shell at $x = d$.



(*c*) (7 Pts.) Find all the points (except infinity) on the x axis at which the electric potential is zero.

(*d*) (6 Pts.) Find all the points (except infinity) on the x axis at which the electric field is zero.

Solution:

(*a*) Since the charge at $x = 0$ is larger in magnitude, the electric potential cannot be zero for $x < 0$. For $x > 0$, we have

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{2x} - \frac{1}{x-d}\right) = 0 \;\rightarrow\; \boxed{x = 3d},\ \text{for } x > d\text{, and } V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{2x} - \frac{1}{d-x}\right) = 0 \;\rightarrow\; \boxed{x = \frac{3}{5}d},\ \text{for } 0 < x < d.$$

(*b*) The electric field can only be zero for $x > d$. $E = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{2x^2} - \frac{1}{(x-d)^2}\right) = 0 \;\rightarrow\; \boxed{x = \frac{\sqrt{3}\,d}{\sqrt{3}-\sqrt{2}} = \left(3+\sqrt{6}\right)d}$

(*c*) In this case the potential on the *x* - axis outside the spherical shell is the same as that in part (*a*). Hence

$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{2x} - \frac{1}{x-d}\right) = 0 \;\rightarrow\; \boxed{x = 3d}$, for $x > d$. The potential inside the shell due to the shell is constant

$V_{shell} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{4d}\right)$. Therefore, $V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{3}{4d}\right) - \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|x-d|}\right) = 0$ gives $|x-d| = \frac{4}{3}d \;\rightarrow\; x = -\frac{d}{3},\ x = \frac{7d}{3}$. Since $\frac{7d}{3} > 2d$, i.e., outside the shell, the answer is $\boxed{x = -\frac{d}{3}}$.

(*d*) Electric field on the positive *x*- axis for $x > 2d$ is the same as that in part (*b*). Therefore it is zero at $\boxed{x = \left(3+\sqrt{6}\right)d}$.

The electric field inside the shell due to the shell is zero. Therefore, the total electric field cannot be zero anywhere inside the shell.

4. Two identical capacitors are connected in parallel and each acquires a charge Q_0 when connected to a source of voltage V_0. The voltage source is disconnected and then a dielectric with $K = 3$ is inserted to fill the space between the plates of one of the capacitors. Determine (*a*) (13 Pts.) the charge now on each capacitor, and (*b*) (12 Pts.) the voltage now across each capacitor.

Solution:

(*a*) Since the capacitors each have the same charge and the same voltage in the initial situation, each has the same capacitance of $C = \frac{Q_0}{V_0}$. When the dielectric is inserted, the total charge of $2Q_0$ will not change, but the charge will no longer be divided equally between the two capacitors. Some charge will move from the capacitor without the dielectric (C_1) to the capacitor with the dielectric (C_2). Since the capacitors are in parallel, their voltages will be the same.

$$V_1 = V_2 \quad \rightarrow \quad \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \quad \rightarrow \quad \frac{Q_1}{C} = \frac{2Q_0 - Q_1}{KC} \quad \rightarrow$$

$$Q_1 = \frac{2}{(K+1)} Q_0 = \frac{2}{4} Q_0 = \boxed{0.5Q_0} \;\; ; \;\; Q_2 = \boxed{1.5Q_0}$$

(*b*) $V_1 = V_2 = \frac{Q_1}{C_1} = \frac{0.5Q_0}{Q_0/V_0} = \boxed{0.5V_0} = \frac{Q_2}{C_2} = \frac{1.5Q_0}{3Q_0/V_0}$.