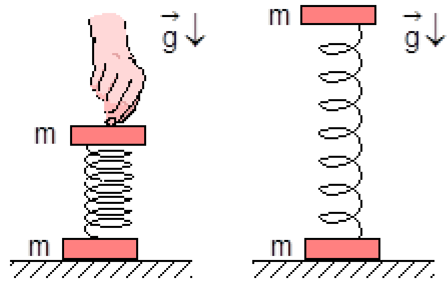
PHYS 101 – General Physics I Final Exam

Duration: 120 minutes Monday, 22 May 2017, 18:30

1. A toy is made from two identical masses m connected by a spring of spring constant k and relaxed length ℓ. Initially, both masses are motionless, one resting on the floor, and the other resting above it.

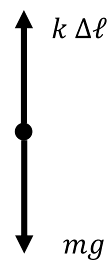
(*a*) (8 Pts.) What is the distance between the masses?

(*b*) (17 Pts.) If we press on the upper mass and suddenly pull our hand back the whole toy can jump off the ground. What should be the maximum distance between the masses before we let go so that the lower mass can leave the table? Is there any other condition?



Solution:

(*a*) From the free body diagram we have $k\,\Delta\ell = mg \;\;\rightarrow\;\; \Delta\ell = \dfrac{mg}{k}$, where $\Delta\ell$ is the amount the spring is compressed. Therefore the distance between the two masses is $h = \ell - \dfrac{mg}{k}$.



(*b*) Suppose we press on the upper mass so that the distance between the masses is now h_0 . Total mechanical energy of the system at this compressed position is

$$E_i = mgh_0 + \frac{1}{2}k(\ell - h_0)^2 .$$

When we let go the compressed spring will push the upper mass until the distance between the masses is ℓ' (note that $\ell' > \ell$). To lift the lower mass, we need to have

$$k(\ell' - \ell) = mg \;\;\rightarrow\;\; (\ell' - \ell) = \frac{mg}{k} .$$

The total mechanical energy at this position will be

$$E_f = mg\ell' + \frac{1}{2}k(\ell' - \ell)^2 .$$

Since total mechanical energy is conserved

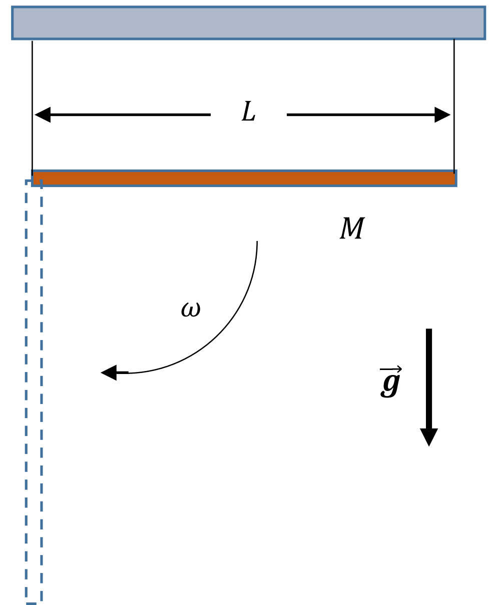
$$mg\ell' + \frac{1}{2}k(\ell' - \ell)^2 = E_i = mgh_0 + \frac{1}{2}k(\ell - h_0)^2$$ which, using $(\ell' - \ell) = \dfrac{mg}{k}$, becomes

$$\frac{k}{2}(\ell - h_0)^2 - mg(\ell - h_0) - \frac{3}{2}\frac{m^2 g^2}{k} = 0 .$$

The positive root of this quadratic equation gives $h_0 = \ell - \dfrac{3mg}{k}$. Since we must have $h_0 > 0$, we need to have $\ell > \dfrac{3mg}{k}$ for the toy to jump.

2. A uniform thin bar of length L and mass M hangs horizontally, suspended from the ceiling by two light massless strings tied to each end, as shown in the figure. Suddenly, one of the strings breaks, and the rod starts to rotate about the other end under the action of the constant gravitational force.

Assume that the unbroken string always remains vertical. $I_{cm} = \dfrac{ML^2}{12}$ for the uniform thin bar



(a) (5 Pts.) Find the tensions in the strings before one of them breaks.

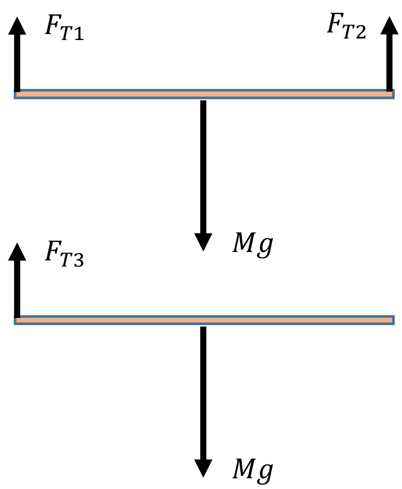
(b) (5 Pts.) Find the angular acceleration of the rod just after the string breaks.

(c) (5 Pts.) Find the tension in the remaining string just after the other one breaks.

(d) (5 Pts.) Find the angular velocity of the rod when it reaches the vertical position.

(e) (5 Pts.) Find the linear velocity of the lower tip of the rod when the rod reaches the vertical position.

Solution:



(a) From symmetry $F_{T1} = F_{T2} = F_T \;\rightarrow\; 2F_T = Mg$

Therefore $F_T = \dfrac{Mg}{2}$.

(b) When the right string breaks, we have

$$\tau = I\alpha \;\rightarrow\; Mg\frac{L}{2} = \left(\frac{1}{3}ML^2\right)\alpha \;\rightarrow\; \alpha = \frac{3g}{2L}.$$

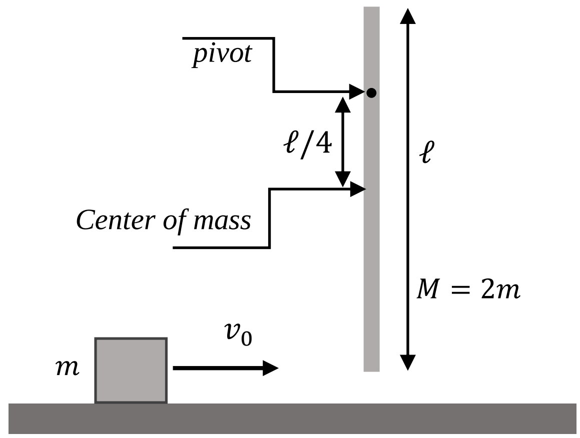
(c) $F = Ma_{CM} \;\rightarrow\; Mg - F_{T3} = Ma_{CM}$. Since $a_{CM} = \dfrac{L}{2}\alpha = \dfrac{3g}{4}$.

So $F_{T3} = Mg - Ma_{CM} = \dfrac{1}{4}Mg$.

(d) Since total mechanical energy is conserved, we have $\Delta E = \dfrac{1}{2}\left(\dfrac{1}{3}ML^2\right)\omega^2 - Mg\dfrac{L}{2} = 0 \;\rightarrow\; \omega = \sqrt{\dfrac{3g}{L}}$.

(e) $v = L\omega = \sqrt{3gL}$.

3. A block of mass m is sliding across a frictionless horizontal surface at speed v_0. It collides with a thin uniform rod of length ℓ and mass $M = 2m$. The rod is pivoted about a frictionless axle through a point which is at a distance $\ell/4$ above its center of mass, and is initially at rest hanging down vertically straight. After the collision, the block moves straight ahead with **half its initial speed** (i.e., $v_0 / 2$). ($I_{cm} = \dfrac{ML^2}{12}$ for the uniform thin bar)



(*a*) (5 Pts.) Which physical quantity or quantities are conserved during this collision?

(*b*) (10 Pts.) What is the rod's angular velocity right after the collision?

(*c*) (10 Pts.) What must be the minimum initial speed v_0 of the block so that the rod rotates through a full cycle?

Solution:

(*a*) Angular momentum of the system about the pivot is conserved.

(*b*) $L_i = mv_0 \dfrac{3\ell}{4}$, $L_f = I_p\omega + m\dfrac{v_0}{2}\dfrac{3\ell}{4}$.

Moment of inertia of the rod around the pivot is found by using the parallel axis theorem.

$$I_p = I_{CM} + 2m\left(\frac{\ell}{4}\right)^2 = \frac{1}{12}(2m)\ell^2 + 2m\left(\frac{\ell}{4}\right)^2 = \frac{7}{24}m\ell^2 .$$

$$mv_0\frac{3\ell}{4} = \frac{7}{24}m\ell^2\omega + m\frac{v_0}{2}\frac{3\ell}{4} \quad \rightarrow \quad \omega = \frac{9v_0}{7\ell} .$$

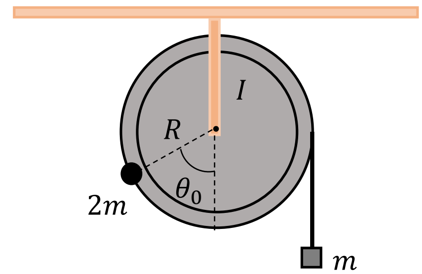
(*c*) After the collision total mechanical energy of the rotating rod is conserved. Therefore

$$\Delta E = 2mg\frac{\ell}{2} - \frac{1}{2}\left(\frac{7}{24}m\ell^2\right)\left(\frac{9v_0}{7\ell}\right)^2 = 0 ,$$

which gives

$$v_{0\,\min} = \frac{4\sqrt{7}}{3\sqrt{3}}\sqrt{g\ell} .$$

4. Consider the system illustrated in the figure. The pulley, with radius R and moment of inertia I, is mounted on a frictionless axle fixed to the ceiling and an object with mass $2m$ is glued to its rim. A chord with negligible mass is wrapped around the pulley and supports an object with mass m. The chord does not slip on the pulley. Initially, the system is at rest in the equilibrium position shown in the figure.



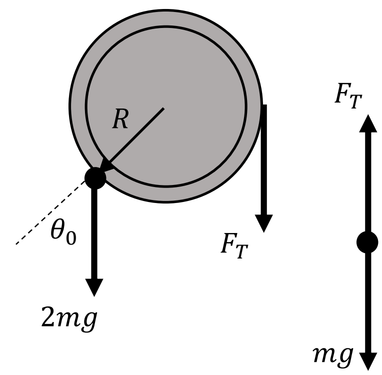
(a) (10 Pts.) Calculate θ_0.

(b) (15 Pts.) Find the frequency of small oscillations about θ_0.

Solution:

(a) When the system is at rest total torque acting on the pulley and the total force acting on the mass are zero.

Considering the free body diagrams given, we write



$$2mgR\sin\theta_0 = mgR \;\rightarrow\; \sin\theta_0 = \frac{1}{2} \;\rightarrow\; \theta_0 = \frac{\pi}{6}\ .$$

(b) If the angle is slightly changed as $\theta_0 + \delta$, where $\delta > 0$ is small, we have

$$\tau = 2mgR\sin(\theta_0 + \delta) - F_T R = -(I + 2mR^2)\alpha$$

For the mass we have $F_T - mg = ma$. Since the chord does not slip on the pulley $a = R\alpha$, and hence

$$F_T = mg + mR\alpha\ .$$

This means

$$2mgR\sin(\theta_0 + \delta) - (mg + mR\alpha)R = -(I + 2mR^2)\alpha\ .$$

Using the expansion $\sin(\theta_0 + \delta) = \sin\theta_0\cos\delta + \cos\theta_0\sin\delta$ and noting that we have for small angles

$\sin\delta \approx \delta$, $\cos\delta \approx 1$, and furthermore that we have $\sin\theta_0 = \frac{1}{2}$, $\cos\theta_0 = \frac{\sqrt{3}}{2}$, $\alpha = \frac{d^2\theta}{dt^2}$, we get

$$\frac{d^2\theta}{dt^2} + \frac{\sqrt{3}mgR}{I + 3mR^2}\delta = 0\ .$$

We identify $\omega^2 = \frac{\sqrt{3}mgR}{I + 3mR^2} \;\rightarrow\; f = \frac{1}{2\pi}\sqrt{\frac{\sqrt{3}mgR}{I + 3mR^2}}$.