# PHYS 101 – General Physics I Final Exam Solutions

Duration: 120 minutes Thursday, 05 January 2017, 09:30

**1.** The position of a 500-g point object is given (in meters) by the vector
$\vec{\mathbf{r}}(t) = (1+40t)\,\hat{\mathbf{i}} + (20t - 5t^2)\,\hat{\mathbf{j}}\,,(0 \le t \le 4\,s)$, where *t* is in seconds.

(a) (5 Pts.) What is the force $\vec{F}$ acting on the object?

(*b*) (5 Pts.) What is the instantaneous power *P* delivered to this object as a function of time?

(*c*) (5 Pts.) How much work *W* is done on the object between $t = 0$ and $t = 2$ s ?

(*d*) (5 Pts.) Find the angular momentum $\vec{L}$ of the object at time $t = 2$ s with respect to the origin.

(*e*) (5 Pts.) Find the torque $\vec{\tau}$ acting on the object at time $t = 2$ s with respect to the origin.

**Solution:**

(*a*) $\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\dfrac{d^2\vec{\mathbf{r}}}{dt^2}$ . $\dfrac{d\vec{\mathbf{r}}}{dt} = \left[(40)\,\hat{\mathbf{i}} + (20-10t)\,\hat{\mathbf{j}}\right](\text{m/s}) \;\rightarrow\; \dfrac{d^2\vec{\mathbf{r}}}{dt^2} = (-10\,\text{m/s}^2)\,\hat{\mathbf{j}}$

$\vec{\mathbf{F}} = (-5\text{ N})\,\hat{\mathbf{j}}$

Since $m = \dfrac{1}{2}$ kg, we have $\vec{\mathbf{F}} = (-5\text{ N})\,\hat{\mathbf{j}}$

(*b*) Using the definition

$$P = \vec{\mathbf{F}}\cdot\vec{\mathbf{v}} = \vec{\mathbf{F}}\cdot\frac{d\vec{\mathbf{r}}}{dt} = \left[(-5\text{N})\,\hat{\mathbf{j}}\right]\cdot\left[(40)\,\hat{\mathbf{i}} + (20-10t)\,\hat{\mathbf{j}}\right](\text{m/s}) = (-100+50t)\text{ Watt}\,.$$

$P = (-100+50t)\,\text{W}$

(*c*) We use the work – energy principle $W = \Delta K = K_f - K_i$

$$\vec{\mathbf{v}}(t) = \left[(40)\,\hat{\mathbf{i}} + (20-10t)\,\hat{\mathbf{j}}\right](\text{m/s}) \;\rightarrow\; \vec{\mathbf{v}}(0) = \left[40\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}\right](\text{m/s}),\; \vec{\mathbf{v}}(2) = \left[40\,\hat{\mathbf{i}}\right](\text{m/s})$$

$$W = \frac{1}{2}mv^2(2) - \frac{1}{2}mv^2(0) = \frac{1}{4}\left[(40)^2 - (40)^2 - (20)^2\right]\text{ J} = -100\text{ J}$$

$W = -100$ J

Note that one gets the same result by evaluating the integral

$$W = \int_0^2 P(t)\,dt = \int_0^2 (-100+50t)\,dt = \left[-100t + 25t^2\right]_0^2 = -100\,\text{J}$$

(*d*) $\vec{\mathbf{L}}(2) = \vec{\mathbf{r}}(2)\times m\vec{\mathbf{v}} = \left[(81\text{ m})\,\hat{\mathbf{i}} + (20\text{ m})\,\hat{\mathbf{j}}\right]\times\left[(20\,\text{kg}\cdot\text{m/s})\,\hat{\mathbf{i}}\right] \;\rightarrow$

$\vec{\mathbf{L}}(2) = (-400\,\text{kg}\cdot\text{m}^2/\text{s})\,\hat{\mathbf{k}}$

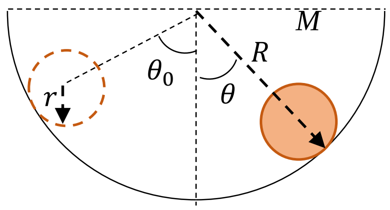
(*e*) $\vec{\boldsymbol{\tau}}(2) = \vec{\mathbf{r}}(2)\times\vec{\mathbf{F}} = \left[(81\text{ m})\,\hat{\mathbf{i}} + (20\text{ m})\,\hat{\mathbf{j}}\right]\times\left[(-5\text{ N})\,\hat{\mathbf{j}}\right] = \;\rightarrow$

$\vec{\boldsymbol{\tau}}(2) = (-405\,\text{m}\cdot\text{N})\,\hat{\mathbf{k}}$

**2.** A cylinder of mass $M$ and radius $r$ rolls without slipping on the inside of a cylindrical track of radius $R$ ($r < R$). The cylinder starts from rest from its initial position where $\theta_0 = -\pi/3$, ($\cos(-\pi/3) = 1/2$), as illustrated in the figure. (Moment of inertia of the cylinder about its symmetry axis is $I = \frac{1}{2}Mr^2$)

(*a*) (12 Pts.) What will be the linear speed $v_C$ of its center when it reaches the lowest point (i.e., $\theta = 0$) of the track?

(b) (13 Pts) Find the angular speed of the cylinder around its axis of rotation as a function of the angle $\theta$.




**Solution:**

(*a*) Rolling means total mechanical energy is conserved. Since initially the cylinder is at rest, its initial total mechanical energy is

$$E_i = M\,g\,(R-r)\left(1-\cos\theta_0\right) = \frac{1}{2}M\,g\,(R-r),$$

where it was assumed that the gravitational potential energy is zero at the lowest position of the center of the cylinder, i.e., when $\theta = 0$. The total mechanical energy at the bottom is

$$E_b = \frac{1}{2}M\,v_C^2 + \frac{1}{2}I\,\omega^2 = \frac{1}{2}M\,v_C^2 + \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\left(\frac{v_C}{r}\right)^2 = \frac{3}{4}M\,v_C^2 .$$

Since energy is conserved, we have

$$E_b = E_i \;\rightarrow\; \frac{3}{4}M\,v_C^2 = \frac{1}{2}M\,g\,(R-r) \;\rightarrow\; v_C = \sqrt{\frac{2}{3}g(R-r)}$$

$$\boxed{v_C = \sqrt{\frac{2}{3}g(R-r)}}$$

(*b*) For an arbitrary value of the angle $\theta$, we have

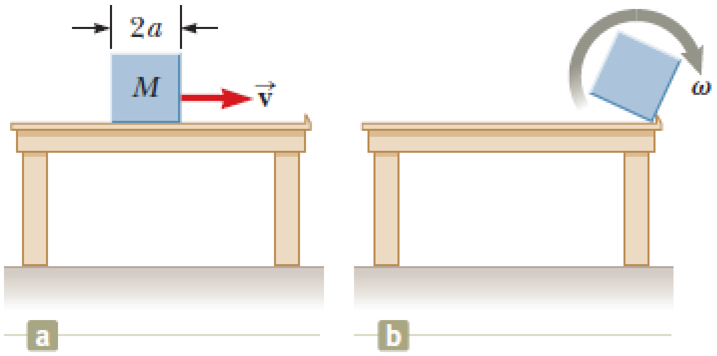
$$K(\theta) = \frac{1}{2}Mv_C^2 + \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\omega^2 = \frac{1}{2}M(r\omega)^2 + \frac{1}{4}Mr^2\omega^2 = \frac{3}{4}Mr^2\omega^2 \text{ , and } U_g = M\,g\,(R-r)\left(1-\cos\theta\right).$$

Since $E(\theta) = E_i$, we have

$$\frac{3}{4}Mr^2\omega^2 + M\,g\,(R-r)\left(1-\cos\theta\right) = \frac{1}{2}M\,g\,(R-r) \;\rightarrow$$

$$\boxed{\omega = \sqrt{\frac{4g(R-r)}{3r^2}\left(\cos\theta - \frac{1}{2}\right)}}$$

**3.** (25 Pts.) A solid cube of side $2a$ and mass $M$ is sliding on a frictionless horizontal surface with constant velocity $\vec{\mathbf{v}}$, as shown in the Figure a. It hits a small obstacle at the end of the table that causes the cube to tilt as shown in Figure b. Find the minimum value of the magnitude of $\vec{\mathbf{v}}$ such that the cube tips over and falls off the table. *Note:* The cube undergoes a completely inelastic collision at the edge. (For the cube of side $2a$ in the problem the center of mass moment of inertia is $I_{CM} = \frac{2}{3}Ma^2$ about an axis perpendicular to any one of the sides.)




**Solution:**

When the cube hits the obstacle at the end of the table, its corner sticks to it so, the collision is completely inelastic. Energy and momentum are not conserved during such a collision because there are external forces acting on the cube at the obstacle. However, angular momentum with respect to the obstacle is conserved, sice these external forces yield no torque about that point.

$L_i = M\,v\,a$ and $L_f = I_e\omega$, where $I_e$ is the moment of inertia, and $\omega$ is the angular speed of the cube just after the collision about its edge . Using the parallel axes theorem, we have

$$I_e = I_{CM} + M\left(a\sqrt{2}\right)^2 = \frac{2}{3}Ma^2 + 2Ma^2 = \frac{8}{3}Ma^2 .$$

Therefore, conservation of angular momentum means

$$M\,v\,a = \frac{8}{3}Ma^2\omega \;\rightarrow\; \omega = \frac{3v}{8a} .$$

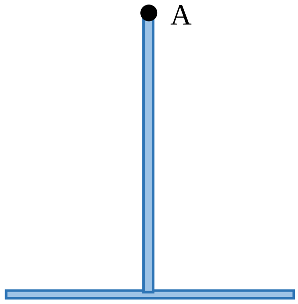
During the course of the motion following the collision, total mechanical energy is conserved. Therefore, for the cube to tip over and fall off the table, its kinetic energy just after the collision must be at least equal to the change in the potential enery, that is

$$\frac{1}{2}\left(\frac{8}{3}Ma^2\right)\left(\frac{3v}{8a}\right)^2 = \frac{3}{16}M\,v^2 \geq M\,g\left(\sqrt{2}-1\right)a .$$

Therefore, the result is:

$$v \geq 4\sqrt{\frac{g\,a}{3}\left(\sqrt{2}-1\right)} .$$

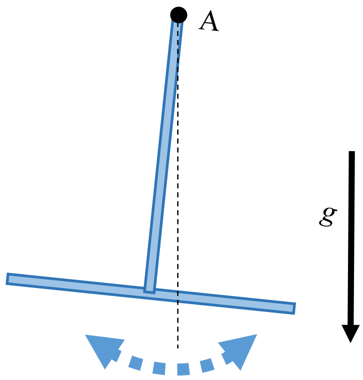
**4.** Two uniform thin rods, each of mass M and length L are connected rigidly to form a T shaped object as shown in the top figure.




($a$) (10 Pts.) Find the distance $d$ of the center of mass of the object from point A.

($b$) (15 Pts.) Find the frequency $f_a$ of small oscillations of this object if it is suspended from point A, and is free to rotate around that point.

(Moment of inertia of a uniform rod of length L and mass M around its center of mass is $I_{CM} = \frac{M\,L^2}{12}$)




**Solution:**

($a$) Since the rods are uniform, their centers of mass are at their geometric centers. Taking point A as the origin and choosing downward direction to be the positive $y$-axis, the center of mass of the T shaped object is found on the $y$-axis at

$$y_{CM} = \frac{1}{2M}\left(M\,\frac{L}{2} + ML\right) = \frac{3}{4}L\,.$$

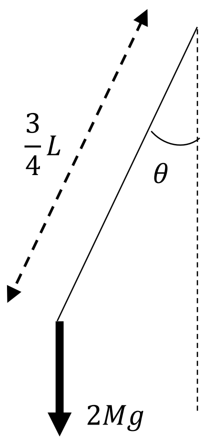
Therefore:

$$d = \frac{3}{4}L$$

($b$) The moment of inertia of the object around the point A is $I_A = I_1 + I_2$, where $I_1$ and $I_2$ are moments of inertia of the two rods around the point A. Using the parallel axes theorem, we have

$$I = \left(\frac{1}{12}ML^2 + M\,\frac{L^2}{4}\right) + \left(\frac{1}{12}ML^2 + ML^2\right) = \frac{17}{12}ML^2\,.$$

From the free body diagram, we see that




$$\tau = I\alpha \;\;\rightarrow\;\; -2M\,g\left(\frac{3}{4}L\right)\sin\theta = \left(\frac{17}{12}ML^2\right)\frac{d^2\theta}{dt^2}\,,$$

which, for small oscillations where $\sin\theta \approx \theta$, is written as

$$\frac{d^2\theta}{dt^2} + \left(\frac{18\,g}{17\,L}\right)\theta = 0\,.$$

We identify the angular frequency $\omega = \sqrt{\frac{18\,g}{17\,L}}$, and hence

$$f_A = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{18\,g}{17\,L}}\,.$$

$$f_A = \frac{1}{2\pi}\sqrt{\frac{18\,g}{17\,L}}$$