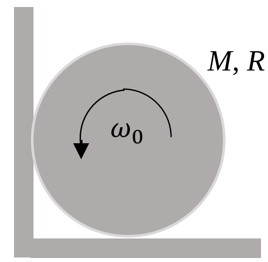
PHYS 101 – General Physics I
Final Exam Solutions, 08.05.2016

1. A cylindrical disk of mass M and radius R ($I_{CM} = \frac{1}{2}MR^2$) skids against both the horizontal and vertical surfaces of a corner as shown in the figure. The coefficient of kinetic friction between the disk and both surfaces is μ_k. The initial angular speed of the disk at the instant it is placed in the corner is ω_0, and its axis of rotation does not move while it rotates.

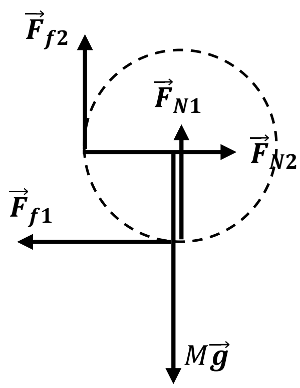


(a) (10 Pts.) Draw a free body diagram for the disk.

(b) (15 Pts.) Find the total number of revolutions the disk makes before stopping.

Solution:

(*a*)



(*b*) We have $\vec{a} = 0$ and $\alpha \neq 0$. Therefore, $\sum \vec{F} = 0$, and $\sum \vec{\tau} \neq 0$. Newton's second law for the translational motion in the horizontal and the vertical directions are written respectively as

$F_{N2} - F_{f1} = 0$, and $F_{N1} + F_{f2} - Mg = 0$. We know that $F_{f1} = \mu_k F_{N1}$ and $F_{f2} = \mu_k F_{N2}$. Hence, we have

$-\mu_k F_{N1} + F_{N2} = 0$ and $F_{N1} + \mu_k F_{N2} = Mg$. Solving these two equations for F_{N1} and F_{N2}, we find

$$F_{N1} = \frac{Mg}{1+\mu_k^2} \text{ and } F_{N2} = \frac{\mu_k Mg}{1+\mu_k^2}\text{. Hence } F_{f1} = \frac{\mu_k Mg}{1+\mu_k^2} \text{ and } F_{f2} = \frac{\mu_k^2 Mg}{1+\mu_k^2}.$$

Newton's second law for the rotational motion is written as $\sum \tau = -RF_{f1} - RF_{f2} = I\alpha$. Hence, the angular acceleration of the disk is found as

$$\alpha = -\frac{R}{I}\left(F_{f1} + F_{f2}\right) = -\frac{2}{MR}\left(\frac{\mu_k Mg}{1+\mu_k^2} + \frac{\mu_k^2 Mg}{1+\mu_k^2}\right) = -\frac{2g}{R}\left(\frac{\mu_k + \mu_k^2}{1+\mu_k^2}\right).$$ Since $\omega^2 - \omega_0^2 = 2\alpha\theta$, and $\omega = 0$ when

the disk stops, we find $\theta = \dfrac{R\omega_0^2(1+\mu_k^2)}{4g(\mu_k + \mu_k^2)}$. Hence, total number of revolutions the disk makes before stopping is

$$\text{Number of Rev} = \frac{\theta}{2\pi} = \frac{R\omega_0^2(1+\mu_k^2)}{8\pi g(\mu_k + \mu_k^2)}.$$

2. The motion of a point particle of mass m is described by the position vector

$\vec{r}(t) = [R\cos(\omega\, t)]\hat{\imath} + [R\sin(\omega\, t)]\hat{\jmath} - R\,\hat{k}$, where R and ω are constants.

(a) (9 Pts) Find the force acting on the particle.

(b) (9 Pts.) Find the angular momentum of the particle.

(c) (7 Pts.) What is the torque acting on the particle?

Solution:

(a) $\vec{\mathbf{v}} = \dfrac{d\vec{\mathbf{r}}}{dt} = -R\omega\sin(\omega t)\,\hat{\mathbf{i}} + R\omega\cos(\omega t)\,\hat{\mathbf{j}}$, and $\vec{\mathbf{a}} = \dfrac{d\vec{\mathbf{v}}}{dt} = -R\omega^2\cos(\omega t)\,\hat{\mathbf{i}} - R\omega^2\sin(\omega t)\,\hat{\mathbf{j}}$.

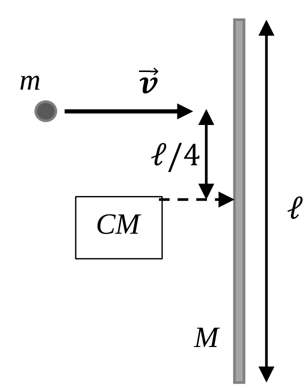
$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = -mR\omega^2\left[\cos(\omega t)\,\hat{\mathbf{i}} + \sin(\omega t)\,\hat{\mathbf{j}}\right]$.

$$(b)\ \vec{\mathbf{L}} = \vec{\mathbf{r}}\times\vec{\mathbf{p}} = \vec{\mathbf{r}}\times m\vec{\mathbf{v}} = mR^2\omega\begin{pmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \cos(\omega t) & \sin(\omega t) & -1\\ -\sin(\omega t) & \cos(\omega t) & 0\end{pmatrix} = mR^2\omega\left[\cos(\omega t)\hat{\mathbf{i}} + \sin(\omega t)\hat{\mathbf{j}} + \hat{\mathbf{k}}\right].$$

$$(c)\ \vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt} = mR^2\omega^2\left[-\sin(\omega t)\hat{\mathbf{i}} + \cos(\omega t)\hat{\mathbf{j}}\right].$$

3. A thin uniform rod of mass $M = 3.0\ kg$ and length $\ell = 1.0\,m$ rests on a frictionless horizontal surface. A plastic puck of mass $m = 0.2\ kg$ moving with speed $v = 10.0\ m/s$ collides with the rod and rebounds with speed $v/2$.

($I_{CM} = \frac{1}{12}ML^2$ for the rod)



(a) (8 Pts.) Find the velocity of the center of mass of the rod after the collision.

(b) (10 Pts.) Find the angular speed of the rod after the collision.

(c) (7 Pts.) What percentage of the initial energy was lost in the collision?

Solution:

(a) Linear momentum is conserved in the collision.

$p_i = mv = (0.2\,kg)(10.0\,m/s) = 2.0\,kg.m/s$, and $p_f = -m\frac{v}{2} + Mv_{cm} = -(0.2\,kg)(5.0\,m/s) + (3.0\,kg)v_{cm}$

$mv = -m\frac{v}{2} + Mv_{cm} \rightarrow v_{cm} = \frac{3mv}{2M} = \frac{3(0.2\,kg)(10.0\,m/s)}{2(3.0\,kg)} = 1.0\,m/s$.

(b) Angular momentum about the center of mass of the rod is conserved in the collision. We have

$L_i = mv\frac{\ell}{4} = (0.2\,kg)(10.0\,m/s)(\frac{1}{4}m) = 0.5\ kg.m/s$,

$I = \frac{1}{12}M\ell^2 = \frac{1}{12}(3.0\,kg)(1m)^2 = 0.25\ kg.m^2$, and

$L_f = -m\frac{v}{2}\frac{\ell}{4} + I\omega = -(0.2\,kg)(5.0\,m/s)(\frac{1}{4}m) + (0.25\ kg.m^2)\omega$.

Since $L_i = L_f$, we have

$mv\frac{\ell}{4} = -m\frac{v}{2}\frac{\ell}{4} + I\omega \rightarrow \omega = \frac{9mv}{2M\ell} = \frac{9(0.2\,kg)(10.0\,m/s)}{2(3.0\,kg)(1m)} = 3.0\,s^{-1}$.

(c) $K_i = \frac{1}{2}mv^2 = \frac{1}{2}(0.2\,kg)(10.0\,m)^2 = 10.0\,J$.

$K_f = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$.

Using the results found, we have

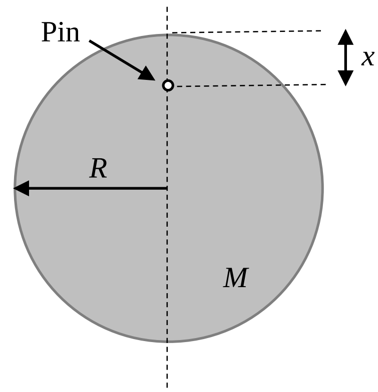
$K_f = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}M\left(\frac{3mv}{2M}\right)^2 + \frac{1}{2}\left(\frac{1}{12}M\ell^2\right)\left(\frac{9mv}{2M\ell}\right)^2 = \left(\frac{1}{4} + \frac{63m}{16M}\right)\frac{1}{2}mv^2$, or

$K_f = \frac{1}{2}(0.2\,kg)(5.0\,m)^2 + \frac{1}{2}(3.0\,kg)(1m/s)^2 + \frac{1}{2}(0.25\,kg.m^2)(3.0\,s^{-1})^2 = 5.1\,J$.

$\frac{\Delta K}{K_i} = \left(\frac{63m}{16M} - \frac{3}{4}\right) = \left(\frac{63(0.2\,kg)}{16(3.0\,kg)} - \frac{3}{4}\right) = -0.49$,

meaning that 49% of the initial energy is lost in the collision.

4. A disk of radius R and mass M has a small hole drilled through it at a distance x from its edge. The disk is hung from the wall by means of a pin through the hole, and used as a pendulum. ($I_{CM} = \frac{1}{2}MR^2$).



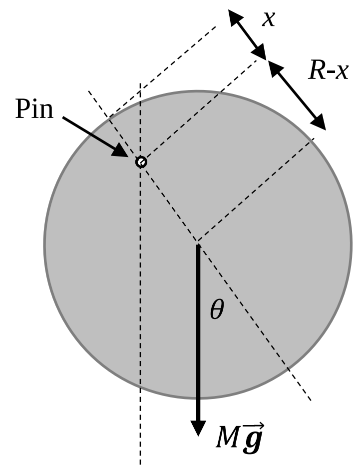
(a) (15 Pts.) What is the period of this pendulum for small oscillations?

(b) (10 Pts.) For what value of x is the period shortest?

Solution:

The moment of inertia of the disk about the pin is found using the parallel axes theorem as

$$I = I_{CM} + M(R-x)^2 = \frac{1}{2}MR^2 + M(R-x)^2 .$$



The net torque acting on the disk about the pin is

$$\tau = -(R-x)Mg\sin\theta .$$

The minus sign is there because the torque tends to decrese the angle θ. Writing Newton's second law for the rotational motion of the disk, we have

$\tau = I\alpha \ \rightarrow \ \dfrac{d^2\theta}{dt^2} = -\dfrac{1}{I}(R-x)Mg\sin\theta$, which, for small oscillations where $\sin\theta \approx \theta$, becomes

$\dfrac{d^2\theta}{dt^2} + \dfrac{2(R-x)g}{R^2+2(R-x)^2}\theta = 0$. We identify $\omega^2 = \dfrac{2(R-x)g}{R^2+2(R-x)^2}$.

The period is found as $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{R^2+2(R-x)^2}{2(R-x)g}}$.

(b) As $x \to R$ we have $T \to \infty$. When $x = 0$. We have $T = 2\pi\sqrt{\dfrac{3R}{2g}}$. For the minimum period we need to have

$\dfrac{dT}{dx} = 2\pi\left(\dfrac{R^2+2(R-x)^2}{2(R-x)g}\right)^{-1/2}\left(\dfrac{R^2-2(R-x)^2}{2g(R-x)^2}\right) = 0$. Solving this equation for x, we find

$x = \left(\dfrac{\sqrt{2}\pm1}{\sqrt{2}}\right)R$. Since $\left(\dfrac{\sqrt{2}+1}{\sqrt{2}}\right)R > R$, the period becomes minimum for $x = \left(\dfrac{\sqrt{2}-1}{\sqrt{2}}\right)R$.

The minimum period is $T_{\min} = 2\pi\sqrt{\dfrac{\sqrt{2}R}{g}}$.