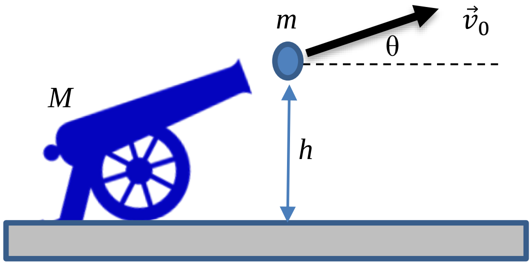
1. A cannon of mass $M = 200\ kg$ which is at rest fires a cannon ball with an initial speed $v_0 = 75\ m/s$ with respect to the ground at an angle θ with the horizontal, as shown in the figure. The mass of the cannon ball is $m = 10\ kg$, and the muzzle of the cannon is $h = 1\ m$ above the ground. (Take $g = 10\ m/s^2$ and $cos\theta = 4/5$.)



(a) (5 Pts.) Find the recoil velocity of the cannon.

(b) (5 Pts.) It is given that the cannon stops moving 1 s after it is fired. Assuming a constant friction force, find the coefficient of friction between the ground and the cannon.

(c) (5 Pts.) Approximately how many seconds (approximate to the nearest integer) after it is fired does the cannon ball hit the ground?

(d) (5 Pts.) What is the distance between the cannon and the cannon ball when it hits the ground?

Solution: (a) Momentum in the horizontal direction is conserved. Therefore

$$Mv_r' + mv_0\cos\theta = 0 \;\rightarrow\; v_r' = \frac{-m}{M}v_0\cos\theta \;\rightarrow\; v_r' = \frac{-10\,kg}{200\,kg}(75\,m/s)\frac{4}{5}$$

$$v_r' = -3\,m/s$$

$\acute{v}_r = -3\ m/s$

(b) The cannon starts to move with initial velocity of $-3\ m/s$ and stops after 1 s. Since

$v(t) = v_0 + at$, we have $0 = (-3\,m/s) + a(1s) \;\rightarrow\; a = 3\,m/s^2$. Since

$F = Ma = (200\,kg)(3\,m/s^2) = 600\,N$. This force is due to kinetic friction, so

$\mu_k = 0.3$

$$F = \mu_k F_N = \mu_k mg \;\rightarrow\; \mu_k = \frac{F}{mg} = \frac{600\,N}{(200\,kg)(10\,m/s^2)} = 0.3 .$$

$t_2 = 9\ s$

(c) Equations describing the motion of the canon ball are

$x(t) = x_0 + v_0\cos\theta,\; y(t) = y_0 + v_0\sin\theta + \frac{1}{2}at^2$. Using $x_0 = y_0 = 0,\ a = -10\ m/s$,

$v_0 cos\theta = (75\ m/s)\left(\frac{4}{5}\right) = 60\ m/s$ and $v_0 sin\theta = (75\ m/s)\left(\frac{3}{5}\right) = 45\ m/s$, we have

$x(t) = (60\,m/s)t,\;\; y(t) = (1m) + (45\,m/s)t - (5\,m/s^2)t^2$. When the cannon ball hits the ground

$y(t) = 0 \;\rightarrow\; 1 + 45t - 5t^2 = 0$, whose solutions are $t_{1,2} = \left(9 \pm \sqrt{81 + \frac{4}{5}}\right)/2$. The positive root

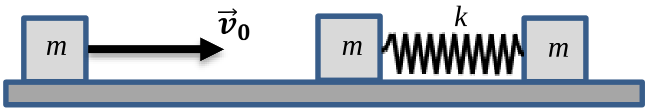
approximated to the nearest integer is $t_2 = 9\,s$.

(d) The cannon ball hits the ground at the point $x(9) = 540\,m\,, y \approx 0$. On the other hand, the cannon moves back the distance $x_{can} = (-3\,m/s)(1s) + \frac{1}{2}(3\,m/s^2)(1s)^2 = -1.5\,m$.

Therefore, the distance between the cannonand the cannon ball is 541.5 m.

$$d = 541.5\,m$$

2. Two blocks of equal mass m, resting on a horizontal surface, are connected by a spring whose spring constant is k. A third block of the same mass m, moving with speed v_0, collides with one of the blocks and instantly sticks to it.



(a) (10 Pts.) Find the speed of the center of mass of the three block system after the collision.

(b) (10 Pts.) Find the maximum compression of the spring.

Solution:

(a) Velocity of the center of mass does not change during the collision. Therefore

$$v_{CM} = v_0/3$$

$V_{CM} = \dfrac{mv_0}{m+2m} = \dfrac{v_0}{3}$, both before and after the collision.

(b) Momentum is conserved during the collision. Consider the instant immediately after the collision, when the spring is not compressed yet and the second mass on the other end of the spring is still at rest.

$mv_0 = (m+m)v' \;\rightarrow\; v' = \dfrac{v_0}{2}$ is the speed of the two stuck masses immediately after the collision.

Hence the total mechanical energy immediately after the collision is

$E = \dfrac{1}{2}(2m)\left(\dfrac{v_0}{2}\right)^2 = \dfrac{1}{4}mv_0^2$. During the instant of maximum compression, the total mechanical energy will be the sum of the potential energy stored in the spring plus the kinetic energy associated with the motion of the center of mass. i.e., $E = \frac{1}{2}(3m)v_{CM}^2 + \frac{1}{2}k\,x_{\max}^2$. After the collision total mechanical energy is conserved. So we have $\frac{1}{2}(3m)\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}k\,x_{\max}^2 = \frac{1}{4}mv_0^2 \;\rightarrow\; x_{\max} = \sqrt{\frac{mv_0^2}{6k}}$.

$$x_{\max} = \sqrt{\frac{mv_0^2}{6k}}$$

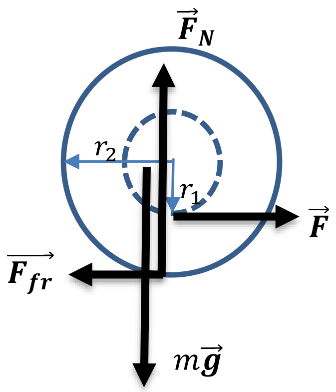
3. A spool of thread (a yo-yo) with inner radius r_1, outer radius r_2, mass M and moment of inertia I about the center of mass is placed on a horizontal surface. The spool is pulled by a thread, as shown in the figure, by a constant force $\vec{F}$. Assume that the spool rolls to the right without slipping.

(a) (6 Pts.) Draw the free-body diagram.

(b) (7 Pts.) Find the linear acceleration of the center of the spool.

(c) (7 Pts.) What is the maximum value of F for which the spool rolls without slipping?

Solution: (a)



$$a_c = \frac{(r_2^2 - r_1 r_2)F}{I + Mr_2^2}$$

$$F_{\max} = \left(\frac{I + Mr_2^2}{I + Mr_1 r_2}\right)\mu_s Mg$$

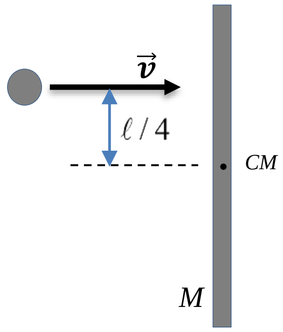
(b) Newton's second law applied to the horizontal translational motion implies $F - F_{fr} = Ma_c$.

Since the spool rolls without slipping, Newton's second law applied to the rotational motion about the center of the spool implies $r_2 F_{fr} - r_1 F = I\alpha = I\frac{a_c}{r_2}$. Solving these two equations for the acceleration, we get $F = \left(\frac{I + Mr_2^2}{r_2^2 - r_1 r_2}\right)a_c$, $F_{fr} = \left(\frac{I + Mr_1 r_2}{r_2^2 - r_1 r_2}\right)a_c$. Hence, we have $a_c = \left(\frac{r_2^2 - r_1 r_2}{I + Mr_2^2}\right)F$.

(c) Since $F_{fr} \le \mu_s Mg$, we have $F_{fr} = \left(\frac{I + Mr_1 r_2}{r_2^2 - r_1 r_2}\right)a_c \le \mu_s Mg \;\rightarrow\; \left(\frac{I + Mr_1 r_2}{I + Mr_2^2}\right)F \le \mu_s Mg$, meaning that $F \le \left(\frac{I + Mr_2^2}{I + Mr_1 r_2}\right)\mu_s Mg$.

4. A thin rod of mass M and length ℓ rests on a frictionles icy surface and is struck at a point $\ell/4$ from its CM by a hockey puck of mass m moving at speed v. It is observed that after the collision the puck recoils with a speed $v/2$ while the rod starts to move to the right rotating about its center of mass at the same time.

($I_{CM} = \frac{1}{12} M\ell^2$ for the rod)



(a) (7 Pts.) Find the velocity of the center of mass of the rod after the collision.

(b) (7 Pts.) Find the angular velocity of the rod about its center of mass after the collision.

(c) (6 Pts.) Find the change in the energy of the system during the collision.

Solution: Linear momentum and angular momentum about the center of mass of the of the rod is conserved during the collision. (a) Conservation of linear momentum implies

$$mv = -m\frac{v}{2} + MV_{CM} \rightarrow V_{CM} = \frac{3mv}{2M} .$$

$$\boxed{V_{CM} = \frac{3mv}{2M}}$$

(b) Conservation of angular momentum about the CM of the rod implies

$$mv\frac{\ell}{4} = -m\frac{v}{2}\frac{\ell}{4} + \frac{1}{12}M\ell^2\omega \rightarrow \omega = \frac{9mv}{2M\ell} .$$

$$\boxed{\omega = \frac{9mv}{2M\ell}}$$

(c) Kinetic energy of the system before the collision is $K_i = \frac{1}{2}mv^2$. After the collision both the recoiling puck and the rod has kinetic energy. Kinetic energy of the rod has both translartional and rotational parts. Hence $K_f = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$. Using the results found, we have

$$K_f = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}M\left(\frac{3mv}{2M}\right)^2 + \frac{1}{2}\left(\frac{1}{12}M\ell^2\right)\left(\frac{9mv}{2M\ell}\right)^2 \rightarrow K_f = \frac{1}{8}(1+\frac{63m}{4M})mv^2 .$$

Hence, $\Delta K = \frac{1}{8}(1+\frac{63m}{4M})mv^2 - \frac{1}{2}mv^2 = -\frac{3}{8}\left(1-\frac{21m}{4M}\right)mv^2$.

$$\boxed{\Delta K = -\frac{3}{8}\left(1-\frac{21m}{4M}\right)mv^2}$$

5. The oscillation of a simple pendulum is given by the equation $\theta(t) = \left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{2}t + \pi\right)$.

(a) (4 Pts.) What is the maximum angular displacement?

(b) (4 Pts.) What is the period of the oscillation?

(c) (4 Pts.) Taking $g = 10\ m/s$, find the length of the pendulum.

(d) (4 Pts.) Find the maximum speed of the pendulum.

(e) (4 Pts.) If the mass of the oscillating object is $m = 0.5\ kg$, what is the total energy?

Solution: Comparing the given equation with the general solution

$\theta(t) = \theta_{\max}\cos(\omega t + \phi)$, where the angular frequency is $\omega = \sqrt{\frac{g}{\ell}}$, we see that

$$\boxed{\theta_{\max} = \frac{\pi}{10}}$$

$\theta_{\max} = \frac{\pi}{10}$, and $\omega = \frac{\pi}{2}s^{-1} \rightarrow \frac{2\pi}{T} = \frac{\pi}{2}s^{-1} \rightarrow T = 4\,s$.

$$\boxed{T = 4\ s}$$

Using $\omega = \sqrt{\frac{g}{\ell}}$, we find $\sqrt{\frac{g}{\ell}} = \frac{\pi}{2} \rightarrow \ell = \frac{40}{\pi^2}m \approx 4m$.

$$\boxed{\ell = \frac{40}{\pi^2}m \approx 4m}$$

The angular speed of the pendulum is

$$\frac{d\theta}{dt} = -\left(\frac{\pi}{10}\right)\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}t + \pi\right) = -\frac{\pi^2}{20}\sin\left(\frac{\pi}{2}t + \pi\right).$$

Maximum angular speed is seen to be $\frac{\pi^2}{20}rad/s$. Since $v_{\max} = \ell\,\omega_{\max}$, we have

$$v_{\max} = \left(\frac{40}{\pi^2}m\right)\left(\frac{\pi^2}{20}s^{-1}\right) = 2\,m/s.$$

$$\boxed{v_{max} = 2\ m/s}$$

The total energy is $E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}\left(\frac{1}{2}kg\right)(2\,m/s)^2 = 1J$.

$$\boxed{E = 1\,J}$$