4. Find the exact values of the sums of the following series.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(n^2-1)^2}$

$$\frac{n}{(n-1)^2} = \frac{1}{4}\cdot\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right)$$

$$\Rightarrow 4S_n = \left(\frac{1}{1^2} - \frac{1}{3^2}\right) + \left(\frac{1}{2^2} - \frac{1}{4^2}\right) + \left(\frac{1}{3^2} - \frac{1}{5^2}\right) + \left(\frac{1}{4^2} - \frac{1}{6^2}\right) + \cdots$$

$$+ \left(\frac{1}{(n-3)^2} - \frac{1}{(n-1)^2}\right) + \left(\frac{1}{(n-2)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right)$$

$$= 1 + \frac{1}{4} - \frac{1}{n^2} - \frac{1}{(n+1)^2} \quad \text{for } n \geq 2.$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{4}\left(1 + \frac{1}{4} - \frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{5}{16}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{n}{(n^2-1)^2} = \frac{5}{16}$$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1 \xrightarrow{d/dx} \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = e^x \xrightarrow{\times x} \sum_{n=1}^{\infty} \frac{n x^n}{n!} = x e^x$$

$$\xrightarrow{d/dx} \sum_{n=1}^{\infty} \frac{n^2 x^{n-1}}{n!} = e^x + x e^x \xrightarrow{\times x} \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} = x e^x + x^2 e^x$$

$$\xrightarrow{d/dx} \sum_{n=1}^{\infty} \frac{n^3 x^{n-1}}{n!} = e^x + x e^x + 2x e^x + x^2 e^x \xrightarrow{(x=1)} \sum_{n=1}^{\infty} \frac{n^3}{n!} = e + e + 2e + e = 5e$$