1. A sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{na_n}$ for $n \geq 1$.

a. Fill in the boxes. No explanation is required. No partial points will be given.

$a_2 = \boxed{2}$ $\quad a_3 = \boxed{\frac{9}{4}}$ $\quad a_4 = \boxed{\frac{259}{108}}$

b. Is the sequence $\{a_n\}_{n=1}^{\infty}$ increasing? ☒ YES ☐ NO

c. Prove the statement you chose in **Part b**.

⊛ For $n=1$, $a_1 = 1 > 0$.

⊛ Suppose $a_k > 0$ for some $k \geq 1$.

Then $a_{k+1} = a_k + \dfrac{1}{k \cdot a_k} > 0 + 0 = 0$.

Hence, by induction, $a_n > 0$ for all $n \geq 1$.

Therefore, $a_{n+1} = a_n + \dfrac{1}{na_n} > a_n + 0 = a_n$ for all $n \geq 1$.

d. Is the sequence $\{a_n\}_{n=1}^{\infty}$ bounded above? ☐ YES ☒ NO

e. Prove the statement you chose in **Part d**.

Since $\{a_n\}_{n=1}^{\infty}$ is increasing, $a_{k+1} = a_k + \dfrac{1}{k \cdot a_k} \Rightarrow a_k a_{k+1} = a_k^2 + \dfrac{1}{k}$

$\Rightarrow a_{k+1}^2 > a_k^2 + \dfrac{1}{k} \Rightarrow a_{k+1}^2 - a_k^2 > \dfrac{1}{k}$ for $k \geq 1$.

Adding these inequalities for $1 \leq k \leq n-1$, we have:

$$a_n^2 - a_1^2 > 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \quad \text{for all } n \geq 2.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (the harmonic series),

$\lim\limits_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) = \infty$. Hence $\lim\limits_{n\to\infty} a_n = \infty$.