5. The position vector of a point P as a function of time t is given by

$$\mathbf{r}(t) = \overrightarrow{OP}(t) = at\,\mathbf{i} + t^2\,\mathbf{j} + (t^3+t)\,\mathbf{k} \quad (-\infty < t < \infty)$$

where a is a constant.

a. Compute the velocity vector $\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t)$.

$\vec{v} = a\vec{i} + 2t\vec{j} + (3t^2+1)\vec{k}$

b. Suppose that the velocity vector $\mathbf{v}$ is parallel to the plane

$$\mathscr{P} : 3x + y + 4z = 1$$

at time $t = -2$. Find all other times t when the velocity vector $\mathbf{v}$ is parallel to this plane.

$\vec{n} = 3\vec{i} + \vec{j} + 4\vec{k}$ is a normal vector for $\mathscr{P}$.

$\vec{v} \parallel \mathscr{P} \iff \vec{v} \perp \vec{n} \iff \vec{v}\cdot\vec{n} = 0 \iff 3a + 1\cdot 2t + 4\cdot(3t^2+1) = 0$

$\vec{v}\big|_{t=-2} \parallel \mathscr{P} \Rightarrow 3a + 1\cdot 2\cdot(-2) + 4\cdot(3\cdot(-2)^2+1) = 0 \Rightarrow a = -16$

Hence $\vec{v} \parallel \mathscr{P} \iff 3\cdot(-16) + 1\cdot 2t + 4\cdot(3t^2+1) = 0 \iff 6t^2 + t - 22 = 0$

$\iff (t+2)(6t-11) = 0 \iff t = -2$ or $t = \frac{11}{6}$

$t = \frac{11}{6}$ is the only other time when $\vec{v} \parallel \mathscr{P}$

c. Suppose that there is a time t when the velocity vector $\mathbf{v}$ is parallel to the line

$$L : \frac{x-1}{5} = \frac{y-2}{-1} = \frac{z+1}{2}.$$

Find all values of the constant a that makes this possible.

$\vec{w} = 5\vec{i} - \vec{j} + 2\vec{k}$ is parallel to L.

$\vec{v} \parallel L \iff \vec{v} \parallel \vec{w} \iff \frac{a}{5} \underset{①}{=} \frac{2t}{-1} \underset{②}{=} \frac{3t^2+1}{2}$

② $\iff 3t^2 + 4t + 1 = 0 \iff (t+1)(3t+1) = 0 \iff t = -1$ or $t = -\frac{1}{3}$

In these cases, ① is also satisfied exactly when $a = 10$ and $a = \frac{10}{3}$, respectively.