3. Find and classify the critical points of $f(x,y) = 2xy^2 - 3x^2 - y^2 + x$.

① $f_x = 2y^2 - 6x + 1 = 0$

② $f_y = 4xy - 2y = 0$

② $\Rightarrow 2 \cdot y \cdot (2x-1) = 0 \Rightarrow y = 0$ or $x = \frac{1}{2}$

$y=0$ and ① $\Rightarrow -6x+1=0 \Rightarrow x = \frac{1}{6} \Rightarrow (x,y) = (\frac{1}{6}, 0)$

$x = \frac{1}{2}$ and ① $\Rightarrow 2y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1$ or $y = -1$

$\Downarrow$ $\Downarrow$

$(x,y) = (\frac{1}{2}, 1)$, $(\frac{1}{2}, -1)$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6 & 4y \\ 4y & 4x-2 \end{vmatrix}$$

$$\Delta(\tfrac{1}{6}, 0) = \begin{vmatrix} -6 & 0 \\ 0 & -\frac{4}{3} \end{vmatrix} = -6 \cdot (-\tfrac{4}{3}) - 0^2 = 8 > 0 \text{ and } f_{xx}(\tfrac{1}{6}, 0) = -6 < 0$$

$\Rightarrow f$ has a local max at $(\frac{1}{6}, 0)$.

$$\Delta(\tfrac{1}{2}, 1) = \begin{vmatrix} -6 & 4 \\ 4 & 0 \end{vmatrix} = -6 \cdot 0 - 4^2 = -16 < 0 \Rightarrow f \text{ has a saddle point at } (\tfrac{1}{2}, 1).$$

$$\Delta(\tfrac{1}{2}, -1) = \begin{vmatrix} -6 & -4 \\ -4 & 0 \end{vmatrix} = -6 \cdot 0 - (-4)^2 = -16 < 0 \Rightarrow f \text{ has a saddle point at } (\tfrac{1}{2}, -1).$$

$\Delta > 0$ and $f_{xx} > 0 \implies$ local minimum
$\Delta > 0$ and $f_{xx} < 0 \implies$ local maximum
$\Delta < 0 \implies$ saddle point

where $\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$