Do not forget to write your ***full name*** *and your* ***Bilkent ID number****, and* ***sign*** *on the upper right corner of your paper.*

Midterm Exam Question 1.

Determine the interval of convergence of the power series $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{2^n(\ln n)^2}$.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

$$c_n = \frac{1}{2^n(\ln n)^2} \Rightarrow \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\left|\frac{1}{2^{n+1}\cdot(\ln(n+1))^2}\right|}{\left|\frac{1}{2^n(\ln n)^2}\right|} = \frac{1}{2}\left(\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}\right)^2$$

$$= \frac{1}{2}\left(\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)}\right)^2 \overset{L'H}{=} \frac{1}{2}\left(\lim_{x\to\infty}\frac{1/x}{1/(x+1)}\right)^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2} \Rightarrow R = 2$$

$$x = 2 \Rightarrow \sum_{n=2}^{\infty} \frac{x^n}{2^n\cdot(\ln n)^2} = \sum_{n=2}^{\infty}\frac{2^n}{2^n\cdot(\ln n)^2} = \sum_{n=2}^{\infty}\frac{1}{(\ln n)^2}$$

$$c = \lim_{n\to\infty}\frac{\frac{1}{(\ln n)^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{(\ln n)^2} = \infty$$ by "useful limits".

Since $c = \infty > 0$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series), $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^2}$ diverges.

$$x = -2 \Rightarrow \sum_{n=2}^{\infty}\frac{x^n}{2^n\cdot(\ln n)^2} = \sum_{n=2}^{\infty}\frac{(-2)^n}{2^n\cdot(\ln n)^2} = \sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^2}.$$

Let $b_n = \frac{1}{(\ln n)^2}$. Then: (0) $b_n = \frac{1}{(\ln n)^2} > 0$ for all $n \geq 2$.

(1) $f(x) = \frac{1}{(\ln x)^2} \Rightarrow f'(x) = -\frac{2}{x\cdot(\ln x)^3} < 0$ for $x > 1 \Rightarrow f$ is decreasing on $(1,\infty)$

$\Rightarrow b_n = f(n) > f(n+1) = b_{n+1}$ for $n \geq 2$.

(2) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{(\ln n)^2} = 0$.

Hence, $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^2}$ converges by AST.

The interval of convergence of the power series is $[-2, 2)$.