Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.

Final Exam Question 5.

Evaluate the triple integral

$$\iiint_B \frac{(x^2-y^2)^2}{(x^2+y^2+z^2)^3}\, dV$$

where $B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$.

In this question you might want to use:

$x = r\cos\theta\,,\ y = r\sin\theta\,,\ z = \rho\cos\phi\,,\ r = \rho\sin\phi$

$dx\,dy\,dz = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

$$\iiint_B \frac{(x^2-y^2)^2}{(x^2+y^2+z^2)^3}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{((\rho\sin\phi\cos\theta)^2-(\rho\sin\phi\sin\theta)^2)^2}{(\rho^2)^3}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \cos^2 2\theta\cdot\sin^4\phi\cdot\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\cos^2 2\theta\,d\theta\cdot\int_0^{\pi}\sin^4\phi\cdot\sin\phi\,d\phi\cdot\int_0^1 d\rho$$

$$= \pi \cdot \frac{16}{15} \cdot 1 = \frac{16}{15}\pi \quad \text{as:}$$

$$\int_0^{\pi}\sin^4\phi\cdot\sin\phi\,d\phi = \int_0^{\pi}(\sin^2\phi)^2\cdot\sin\phi\,d\phi = \int_0^{\pi}(1-\cos^2\phi)^2\cdot\sin\phi\,d\phi \overset{\boxed{u=\cos\phi \\ du=-\sin\phi\,d\phi}}{=}$$

$$= \int_1^{-1}(1-u^2)^2(-du) = \int_{-1}^1(1-2u^2+u^4)\,du = \left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]_{-1}^1 = 2\cdot\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16}{15}$$

$$\int_0^{2\pi}\cos^2 2\theta\,d\theta = \int_0^{2\pi}\frac{1+\cos 4\theta}{2}\,d\theta = \left[\frac{1}{2}\theta+\frac{1}{8}\sin 4\theta\right]_0^{2\pi} = \pi$$