Do not forget to write your **full name** and your **Bilkent ID number**, and **sign** on the upper right corner of your paper.

Final Exam Question 1.

A sequence $\{a_n\}_{n=0}^{\infty}$ satisfies the following:

$$a_0 = \frac{\square\square\square\square}{2503}$$

Write the last four digits of your Bilkent ID number here!

$$a_{n+1} = \frac{p_n + 1}{q_n + 1} \quad \text{if } a_n = \frac{p_n}{q_n}$$ where p_n and q_n are positive integers with no common divisor greater than 1, for $n \geq 0$.

a. Find the limit $\lim\limits_{n\to\infty} a_n$.

b. Find the limit $\lim\limits_{n\to\infty} (a_n)^n$.

Show all your work!
Explain your reasoning fully and in detail using correct mathematical notation and terminology, and in well-formed mathematical and English sentences!

Suppose $a_0 > 1$.

(a) Let d_n be the greatest common divisor of p_n+1 and q_n+1. Then $p_{n+1} = \frac{p_n+1}{d_n}$, $q_{n+1} = \frac{q_n+1}{d_n}$, and $p_{n+1} - q_{n+1} = \frac{p_n - q_n}{d_n}$. Hence $\{p_n - q_n\}_{n=0}^{\infty}$ is a nonincreasing sequence of positive integers. Therefore it must be constant from some index N on. That is, $p_n - q_n = c$ and $d_n = 1$ for all $n \geq N$, for some N and c. In particular, $q_n = q_N + n - N$ for $n \geq N$ and $a_n = \frac{q_N + n - N + c}{q_N + n - N}$ for $n \geq N$. Therefore, $a_n = \frac{1 + (q_N - N + c)/n}{1 + (q_N - N)/n} \to 1$ as $n \to \infty$

(b) Observe that c in part a must be 1 as otherwise one of the integers $q_n = q_N + n - N$ for $N \leq n \leq N+c-1$ will be divisible by c and this will make $p_n = q_n + c$ also divisible by c, contradicting $d_n = 1$ for $n \geq N$.

Hence

$$a_n^n = \left(\frac{q_N + n - N + 1}{q_N + n - N}\right)^n = \frac{\left(1 + \frac{q_N - N + 1}{n}\right)^n}{\left(1 + \frac{q_N - N}{n}\right)^n} \to \frac{e^{q_N - N + 1}}{e^{q_N - N}} = e \text{ as } n \to \infty$$

Solutions for $a_0 < 1$ are similar with the answer $\frac{1}{e}$ in part b.