**4.** A sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$a_1 = 1\,,\quad a_2 = A\,,\quad \text{and}\quad a_n = \frac{a_{n-1}+a_{n-2}}{a_{n-1}-a_{n-2}}\cdot a_{n-1}\quad \text{for } n \geq 3\,,$$

where $A$ is a real number such that $A \neq 1$, $A \neq 0$, $A \neq -1$.

**a.** In the following, fill in the $\boxed{\phantom{xx}}$s with real numbers that will make the sentence into a true statement.

If $A = \boxed{2}$, then $a_3 = \boxed{6}$ and $a_4 = \boxed{12}$.

**b.** In each of the following, fill in the $\boxed{\phantom{xx}}$ with a real number that will make the corresponding sentence into a true statement.

☐ ❶ If $A = \boxed{2}$, then $\lim_{n\to\infty} a_n = \infty$.

☐ ❷ If $A = \boxed{-2}$, then $\lim_{n\to\infty} a_n = 0$.

☒ ❸ If $A = \boxed{\sqrt{2}-1}$, then $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} |a_n| \neq \infty$.

**c.** Now choose exactly one of the statements you made in **Part b** by putting a ✗ in the corresponding ☐, and prove it fully and carefully by using correct mathematical reasoning and notation.

If $A = \sqrt{2}-1$, then $a_3 = \frac{\sqrt{2}-1+1}{\sqrt{2}-1-1}\cdot(\sqrt{2}-1) = -1$, $a_4 = \frac{-1+\sqrt{2}-1}{-1-(\sqrt{2}-1)}\cdot(-1) = 1-\sqrt{2}$,

$a_5 = \frac{1-\sqrt{2}+(-1)}{1-\sqrt{2}-(-1)}\cdot(1-\sqrt{2}) = 1$, $a_6 = \frac{1+1-\sqrt{2}}{1-(1-\sqrt{2})}\cdot 1 = \sqrt{2}-1$.

Hence the pattern $1, \sqrt{2}-1, -1, 1-\sqrt{2}, 1, \sqrt{2}-1, \ldots$ repeats.

Therefore $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} |a_n| \neq \infty$.