4. Consider the function $f(x, y) = x^2 y(x^2 + y^2 - 1)$.

a. Find all critical points of $f$. [Do not classify them!]

$$\left.\begin{aligned} f_x &= 2xy(x^2+y^2-1) + x^2y \cdot 2x = 0 \\ f_y &= x^2(x^2+y^2-1) + x^2y \cdot 2y = 0 \end{aligned}\right\} \circledast$$

If $x=0$, then both equations in $\circledast$ are satisfied.

Therefore, every point $(0,y)$ on the $y$-axis is a critical point.

Suppose $x \neq 0$. Then:

$\circledast \iff y(2x^2+y^2-1)=0$ and $x^2+3y^2-1=0$

$\iff (y=0$ or $2x^2+y^2=1)$ and $x^2+3y^2=1$

$\iff (y=0$ and $x^2+3y^2=1)$ or $(2x^2+y^2=1$ and $x^2+3y^2=1)$

$\iff (y=0$ and $x^2=1)$ or $(x^2=\frac{2}{5}$ and $y^2=\frac{1}{5})$

Hence the critical points of $f$ are:

$(1,0)$, $(-1,0)$, $\left(\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{5}}\right)$, $\left(\sqrt{\frac{2}{5}}, -\sqrt{\frac{1}{5}}\right)$, $\left(-\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{5}}\right)$, $\left(-\sqrt{\frac{2}{5}}, -\sqrt{\frac{1}{5}}\right)$,

and $(0,y)$ for all $y$.

b. Choose one of the critical points of $f$ that lies on the $y$-axis by filling in the box:

$$(x, y) = (0, \boxed{2})$$

Determine whether this point is a local maximum, local minimum, or a saddle point without using the 2nd Derivative Test.

If $y>1$, then $x^2 \geq 0$, $y \geq 0$, and $x^2+y^2-1 \geq y^2-1 \geq 0$.

Hence, if $y>1$, then $f(x,y) = x^2y(x^2+y^2-1) \geq 0 = f(0,2)$.

Therefore, $f$ has a local minimum at $(0,2)$.