2. The *Bateman-Burgers equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial u}{\partial x}$$

arises in the study of nonlinear acoustics and gas dynamics in fluid mechanics, and in the study of traffic flow in civil engineering.

Find all possible values of the pair of constants (a, b) for which the function

$$u(x,t) = \frac{x}{ax^2 + bt + 1}$$

satisfies the Bateman-Burgers equation for all (x, t) with $ax^2 + bt + 1 \neq 0$.

$$u_t = -bx \cdot (ax^2+bt+1)^{-2}$$

$$u_x = (ax^2+bt+1)^{-1} - x \cdot 2ax \cdot (ax^2+bt+1)^{-2}$$

$$u_{xx} = -2ax\,(ax^2+bt+1)^{-2} - 4ax\,(ax^2+bt+1)^{-2} + 2x \cdot (2ax)^2 \cdot (ax^2+bt+1)^{-3}$$

Hence:

$u_t = u_{xx} + u \cdot u_x$ for all (x,t) with $ax^2+bt+1 \neq 0$

$$\Updownarrow$$

$$-bx \cdot (ax^2+bt+1)^{-2} = -6ax \cdot (ax^2+bt+1)^{-2} + 8a^2x^3\,(ax^2+bt+1)^{-3}$$
$$+ x \cdot (ax^2+bt+1)^{-2} - 2ax^3 \cdot (ax^2+bt+1)^{-3}$$

for all (x,t) with $ax^2+bt+1 \neq 0$

$$\Updownarrow$$

$(6a-b-1)(ax^2+bt+1) = 2a(4a-1)x^2$ for all (x,t)

$$\Updownarrow$$

$$\left.\begin{cases} 6a-b-1 = 0 \\ (6a-b-1)\cdot b = 0 \\ (6a-b-1)\cdot a = 2a\cdot(4a-1) \end{cases}\right\} \iff 6a-b-1=0 \text{ and } a(4a-1)=0$$

$\iff$ $(a=0$ and $b=-1)$ or $(a=\frac{1}{4}$ and $b=\frac{1}{2})$

$\iff$ $(a,b) = (0,-1)$ or $(\frac{1}{4}, \frac{1}{2})$