1. Consider the function $f(x,y,z) = \dfrac{x}{y} - \dfrac{y}{z}$ and the point $P_0(3,1,1)$.

a. Compute $\nabla f(P_0)$.

$$\vec{\nabla} f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k} = \frac{1}{y}\vec{i} - \left(\frac{x}{y^2}+\frac{1}{z}\right)\vec{j} + \frac{y}{z^2}\vec{k}$$

$$\Rightarrow \vec{\nabla} f(P_0) = \vec{i} - 4\vec{j} + \vec{k}$$

b. Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(P_0) = 5$? If YES, find one. If NO, prove that it does not exist.

No, because $D_{\vec{u}}f(P_0)$ can be at most $|\vec{\nabla} f(P_0)| = \sqrt{1^2+(-4)^2+1^2} = \sqrt{18}$ and $\sqrt{18} < 5$.

c. Is there a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(P_0) = 3$? If YES, find one. If NO, prove that it does not exist.

YES, because if $\vec{u} = \dfrac{2\vec{i} - 2\vec{j} - \vec{k}}{3}$, then

$$D_{\vec{u}}f(P_0) = \vec{\nabla} f(P_0)\cdot\vec{u} = \frac{1\cdot 2 + (-4)\cdot(-2) + 1\cdot(-1)}{3} = 3.$$

d. Let S be the set of all points $P(x,y,z)$ where f increases fastest in the direction of the vector $\mathbf{A} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}$. Show that S is a subset of the union $L_1 \cup L_2$ of two lines L_1 and L_2, and find parametric equations of these lines.

$\Leftrightarrow$ $\vec{\nabla} f = c\cdot\vec{A}$ for some $c>0$ $\Leftrightarrow$ $\dfrac{1/y}{2} = \dfrac{-(x/y^2+1/z)}{1} = \dfrac{y/z^2}{2}$ and $y>0$

$\Leftrightarrow$ $z^2 = y^2$ and $\dfrac{1}{y} = -2\cdot\left(\dfrac{x}{y^2}+\dfrac{1}{z}\right)$ and $y>0$

$\Leftrightarrow$ ($z=y$ and $x = -\frac{3}{2}y$ and $y>0$)

or ($z=-y$ and $x=\frac{1}{2}y$ and $y>0$)

Hence, $S \subset L_1 \cup L_2$ where L_1: $x = -\frac{3}{2}t,\ y=t,\ z=t;\ -\infty<t<\infty$

and L_2: $x = \frac{1}{2}t,\ y=t,\ z=-t;\ -\infty<t<\infty.$