**4.** Evaluate the integral

$$\iint_D (x^2+y^2)^3 \, dA$$

where $D$ is the region bounded by the hyperbolas $x^2-y^2=1$, $x^2-y^2=4$, $xy=1$, and $xy=-1$ in the right half plane.

Let $u = x^2 - y^2$ and $v = xy$. Then:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2\cdot(x^2+y^2) \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2\cdot(x^2+y^2)} \quad \circledast$$

$$(x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2-y^2)^2 + 4\cdot(xy)^2 = u^2 + 4v^2 \quad \circledast$$

$$\iint_D (x^2+y^2)^3 \, dx\,dy = \iint_G \underbrace{(x^2+y^2)^2}_{} \cdot \underbrace{(x^2+y^2)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{} du\,dv$$

$$= \int_{-1}^{1}\int_{1}^{4} (u^2+4v^2)\cdot\frac{1}{2}\,du\,dv = \int_{-1}^{1} \left[\frac{1}{6}u^3 + 2v^2u\right]_{u=1}^{u=4} dv$$

$$= \int_{-1}^{1} \left(\frac{21}{2} + 6v^2\right) dv = \left[\frac{21}{2}v + 2v^3\right]_{-1}^{1} = 21 + 4 = 25$$