2. The *surface area SA* of the graph of a function $f(x, y)$ on a region D is defined by the formula:

$$SA = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

a. Show that this formula gives the surface area of the upper half of the unit sphere correctly by considering the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ on the unit disk $D = \{(x, y) : x^2 + y^2 \le 1\}$.

$$\left.\begin{aligned} f_x &= -\frac{x}{\sqrt{1-x^2-y^2}} \\ f_y &= -\frac{y}{\sqrt{1-x^2-y^2}} \end{aligned}\right\} \Rightarrow 1 + (f_x)^2 + (f_y)^2 = 1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} = \frac{1}{1-x^2-y^2}$$

$$SA = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\, dA = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-\sqrt{1-r^2}\right]_{r=0}^{r=1} d\theta = \int_0^{2\pi} d\theta = 2\pi = \frac{4\pi \cdot 1^2}{2}$$

b. Assume that g is a differentiable function on the interval $[a, b]$ with $0 < a < b$ and consider the function $f(x, y) = g(r)$ on the ring $D = \{(x, y) : a^2 \le x^2 + y^2 \le b^2\}$ where $r = \sqrt{x^2 + y^2}$. Express the surface area SA of the graph of f on D as a definite integral with respect to r whose integrand involves only r and $g'(r)$.

$$\left.\begin{aligned} f_x &= g'(r)\cdot\frac{\partial r}{\partial x} = g'(r)\cdot\frac{x}{\sqrt{x^2+y^2}} = g'(r)\cdot\frac{x}{r} \\ f_y &= g'(r)\cdot\frac{\partial r}{\partial y} = g'(r)\cdot\frac{y}{\sqrt{x^2+y^2}} = g'(r)\cdot\frac{y}{r} \end{aligned}\right\} \Rightarrow 1 + (f_x)^2 + (f_y)^2 = 1 + \frac{x^2}{r^2} g'(r)^2 + \frac{y^2}{r^2} g'(r)^2 = 1 + g'(r)^2$$

$$SA = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \int_a^b \int_0^{2\pi} \sqrt{1 + g'(r)^2}\, r\, d\theta\, dr = 2\pi \int_a^b r \cdot \sqrt{1 + g'(r)^2}\, dr$$